Partial sums of *m*-th powers with Faulhaber polynomials

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In a previous article we saw how to use Faulhaber polynomials *F(m)* to obtain formulas of second partial sums of *m*-th powers.

$$P_2(m) = \sum_{k=1}^{n} F(m) = (n+1)F(m) - F(m+1) \qquad (1)$$

We will seek now an iterative process for deriving, starting from the formula (1), the polynomial expressions that calculate the partial sums later.

Whereas each partial sum represents, by definition, the sum of the first *n* terms of the previous sum, we have, in general:

$$P_j(m) = \sum_{k=1}^{n} P_{j-1}(m)$$

Then, the polynomial expression of the *j*-th partial sum of the *m*-th power is obtained by adding (from 1 to *n*) all terms of the polynomial of the previous sum. In the practice is better to use non factored polynomials, ranked according to *n* powers: to each power of *n* will correspond the *F(m)* polynomial.

To clarify, we will perform the derivation of the polynomial P_3 on the first power:

F(1) = n*(1+n)/2 F(2) = n*(1+n)*(1+2*n)/6 F(3) = (n*(n+1)/2)^2

$P_2(1)$ = (n^3 + 3*n^2 + 2*n)/6

$P_3(1)$ = Σ $P_2(1)$ = Σ (n^3 + 3*n^2 + 2*n)/6 = (F(3) + 3* F(2) + 2* F(1))/6

= ((n*(n+1)/2)^2 + 3* n*(1+n)*(1+2*n)/6 + 2* n*(1+n)/2)/6

= n*(1 + n)*(2 + n)*(3 + n)/24

The knowledge of the general formula (1) permits the *direct derivation* of the *m*-th power polynomials.

With "Mathematica", using mainly *Factor* and *Table* commands and a bit of *copy* and *paste*, you get faster:

m	Third partial sums of m-th powers: $P_3(m)$
1	n*(1 + n)*(2 + n)*(3 + n)/24
2	n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)/120
3	n*(1 + n)*(2 + n)*(3 + n)*(1 + 3*n + n^2)/120
4	n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)*(-1 + 6*n + 2*n^2)/840
5	n*(1 + n)*(2 + n)*(3 + n)*(-1 + 2*n + n^2)*(2 + 4*n + n^2)/336
6	n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)*(2 - 30*n + 35*n^2 + 30*n^3 + 5*n^4)/5040
7	n*(1 + n)*(2 + n)*(3 + n)*(6 - 6*n - 20*n^2 + 15*n^3 + 25*n^4 + 9*n^5 + n^6)/720
8	n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)*(1 + 36*n - 69*n^2 + 45*n^4 + 18*n^5 + 2*n^6)/3960
9	n*(1 + n)*(2 + n)*(3 + n)*(-50 + 84*n + 127*n^2 - 204*n^3 - 97*n^4 + 126*n^5 + 98*n^6 + 24*n^7 + 2*n^8)/2640
10	

m	Fourth partial sums of m-th powers: $P_4(m)$
1	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)/120
2	n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)/360
3	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(2 + 4*n + n^2)/840
4	n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)*(-1 + 12*n + 3*n^2)/5040
5	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(-24 + 20*n + 85*n^2 + 40*n^3 + 5*n^4)/15120
6	n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)*(-1 - 8*n + 14*n^2 + 8*n^3 + n^4)/5040
7	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(48 - 100*n - 89*n^2 + 160*n^3 + 140*n^4 + 36*n^5 + 3*n^6)/23760
8	n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)*(1 + 4*n + n^2)*(21 - 48*n + 20*n^2 + 16*n^3 + 2*n^4)/23760
9	
10	

m	Fifth partial sums of m-th powers: $P_5(m)$
1	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)/720
2	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(5 + 2*n)/5040
3	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(10 + 15*n + 3*n^2)/20160
4	n^2*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)^2*(5 + 2*n)/30240
5	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(-2 + 5*n + n^2)*(9 + 10*n + 2*n^2)/60480
6	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(5 + 2*n)*(-3 + 5*n + n^2)*(4 + 15*n + 3*n^2)/332640
7	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(-3 + 5*n + n^2)*(-2 + 5*n + n^2)*(5 + 5*n + n^2)/95040
8	
9	
10	

m	Sixth partial sums of m-th powers: $P_6(m)$
1	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(6 + n)/5040
2	n*(1 + n)*(2 + n)*(3 + n)^2*(4 + n)*(5 + n)*(6 + n)/20160
3	n*(1 + n)^2*(2 + n)*(3 + n)*(4 + n)*(5 + n)^2*(6 + n)/60480
4	n*(1 + n)*(2 + n)*(3 + n)^2*(4 + n)*(5 + n)*(6 + n)*(1 + 12*n + 2*n^2)/302400
5	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(6 + n)*(-29 + 54*n + 81*n^2 + 24*n^3 + 2*n^4)/665280
6	
7	

m	Seventh partial sums of m-th powers: $P_7(m)$
1	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(6 + n)*(7 + n)/40320
2	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(6 + n)*(7 + n)*(7 + 2*n)/362880
3	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(6 + n)*(7 + n)*(7 + 7*n + n^2)/604800
4	n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(6 + n)*(7 + n)*(7 + 2*n)*(7 + 42*n + 6*n^2)/19958400
5	
6	

And so on ….