A Fibonacci-Tribonacci Fusion Technique

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The Fibonacci sequence is generated by the recursive formula

$$F_0 = 0, \quad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2} \tag{1.1}$$

Iteration of (1.1) gives the familiar 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610… A000045

The tribonacci sequence is generated by the recursive formula

$$T_0 = 0, \quad T_1 = 1, \quad T_2 = 1, \qquad T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{1.2}$$

Iterating (1.2) gives 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768… A000073

Based on a 'conditional coefficients' idea shared by Michael Somos, (1.1) and (1.2) are 'hybridized' in (1.3).

$$H_{-1} = -1, \quad H_0 = 0, \quad H_1 = 1, \qquad H_n = H_{n-1} + bH_{n-2} + cH_{n-3}$$
$$\text{for } n > 1, \text{ if } \begin{cases} H_{n-1} \text{ is odd, then } b = 1, c = 0 \\ H_{n-1} \text{ is even, then } b = 0, c = 1 \end{cases} \tag{1.3}$$

I.e., when F_{n-1} is odd, this sequence is generated by the usual Fibonacci-type recursion: $H_n = H_{n-1} + H_{n-2}$. When H_{n-1} is even, then a tribonacci-type recursion kicks in and $H_n = H_{n-1} + H_{n-3}$.

For n = 1, 2, 3…, (1.3) generates A254308 = 0, 1, 1, 2, 3, 5, 8, 11, 19, 30, 41, 71, 112, 153, 265…

Working backwards for $a_{n<0}$, i.e., terms left of zero: A = …–989, –418, 153, –265, –112, 41, –71, –30, 11, –19, –8, 3, –5, –2, 1, –1, 0, 1, 1, 2, 3, 5, 8, 11, 19, 30, 41, 71, 112, 153, 265, 418, 571, 989, 1560, 2131, 3691, 6822, 8953, 15775, 24728, 33681, 58409, 92090, 125771, 217861, 343632, 569403…

A is factored into its nested sequences:

a_{3n-1} = …–989, –265, –71, –19, –5, –1, 1, 5, 19, 71, 265, 989, 3691, 15775, 58409, 217861… = A001834

a_{3n} = …–418, –112, –30, –8, –2, 0, 2, 8, 30, 112, 418, 1560, 6822, 24728, 92090, 343632… = A052530

a_{3n+1} = …153, 41, 11, 3, 1, 1, 3, 11, 41, 153, 571, 2131, 8953, 33681, 125771, 569403… = A001835

Most Fibonacci identities seem not to work for this sequence, but two that do are these:

$$(i)\quad \frac{a_{3n}}{2} + \frac{a_{3n-3}}{2} = a_{3n-1} \qquad\qquad (ii)\quad \frac{a_{3n}}{2} - \frac{a_{3n-3}}{2} = a_{3n-2}$$