The following theorem relates to the OEIS sequence A254077, which has definition: "a(n) = n if n <= 3, otherwise the smallest number not occurring earlier such that gcd(a(n),a(n-2)) > gcd(a(n),a(n-1))."

In particular, it is based on the unproven premise that the sequence is infinite. Hence the acronym **ITSII** (If the sequence is infinite) that prefixes the Theorem and each step in its proof.

Theorem. ITSII, the sequence is a permutation of the positive integers.

The proof is based on the following steps.

Step 1. ITSII, the sequence cannot degenerate to a series of terms that are all products of (non-trivial) powers of a single finite set of primes. More formally, there exists no finite set of primes P with k elements, p_1 thru p_k, and integer n, such that, for all m>=n, a(m)=prod(p_i^v_i_m) for i=1 thru k and v_i_m > 0.

Step 2. ITSII, for any prime p, there is at least one term having p as a factor.

Step 3. Define "a prime p becomes permanent" to mean that there is some integer n such that, for all i >= n, p divides a(i). Then, ITSII, there is at most 1 prime that becomes permanent. We will see later that there is no such prime.

Step 4. ITSII, there is an infinite number of odd terms.

Step 5. ITSII, there is an infinite number of even terms.

Step 6. ITSII, any prime p is a factor of an infinite number of terms.

Step 7. ITSII, any prime p is a term.

Step 8. ITSII, the sequence cannot degenerate to a series of terms that all have, as a factor, at least one of a single finite set of primes. More formally, there exists no set of primes P with k elements, p_1 thru p_k, and integer n, such that, for all m>=n, there exists i <= k such that p_i divides a(m).

Step 9. ITSII, the sequence is a permutation of the positive integers.

Step 1. ITSII, the sequence cannot degenerate to a series of terms that are all products of (non-trivial) powers of a single finite set of primes. More formally, there exists no finite set of primes P with k elements, p_1 thru p_k, and integer n, such that, for all m>=n, a(m)=prod(p_i^v_i_m) for i=1 thru k and v_i_m > 0.

Proof.

Clearly, this is true for k=1. That is, the sequence cannot degenerate to a series of terms that are all powers of a single prime p. For, given consecutive terms u,v, either u would divide v or vice versa. But by theorems [1] and [2] on OEIS web page, this is impossible, ITSII.

Now prove that, ITSII, the sequence cannot degenerate to a series of terms that are all products of (non-trivial) powers of a single finite set of primes having at least two elements.

Prove by contradiction, first assuming that the sequence does in fact degenerate to a series of terms that are all products of (non-trivial) powers of a single finite set of primes having k elements, where k >= 2.

1. Show that there must be some first prime z that is not a factor of any term of the sequence
2. Show that prime factors of the terms are “fluctuating”, “stable” of “increasing”
3. Show that there must be fluctuating factors
4. Show that the fluctuating factors must be > z
5. Show that there are no stable primes
6. Show that there are no increasing primes
7. Deduce that there are only fluctuating prime factors, and their powers are all > z
8. Show that this leads to a contradiction

1. After sequence degeneration, all terms are the products of the same set of primes P. Define y to be the maximum value present in the union of P and all the prime factors of terms that precede the degeneration. Then define z to be the first prime after y.
2. For each p_i in P, there must be a point in the sequence after which the exponents of p_i are:

- Fluctuating: the exponents may be locally stable, but there are always, through infinity, pairs of consecutive terms that have, for p_i, increasing exponents, AND there are always, through infinity, pairs of consecutive terms that have, for p_i, decreasing exponents.
- Increasing: the exponents may be locally stable, but there are always, through infinity, pairs of consecutive terms that have, for p_i, increasing exponents.
- Stable: after a certain point in the sequence, the exponents stay the same, through infinity.

There cannot be, in the long run, “decreasing” exponents, as these would at some time reach the minimum value 1 and decrease no more.
3. ITSII, from one term to the next, at least one prime factor must have a decreasing exponent, as otherwise the one term would divide the next, terminating the sequence. Thus, as there cannot be permanently decreasing prime factors, there must be at least one fluctuating prime factor.
4. Locate 3 consecutive terms, r,s,t, after the point when degeneration has set in, and such that r,s,t and all successive terms are > z^(k+2). Choose also r,s,t such that, from r to s the exponent of some fluctuating prime p_i decreases.
Define the exponents of p_i in r and s to be e+x and e respectively, for positive integers e and x.
Suppose p_i^(e+1) < z^(k+1); then we could have chosen z.p_i^(e+1) over t, as:

- it is certainly not already in the sequence.
- it is < z^(k+2) and so < t.
- it satisfies the gcd

So p_i^(e+1) >= z^(k+1), and as k >= 2, p_i^(e+1) > z^2.

---

[1] Note that if a(n) ever divides a(n+1), the sequence will terminate.
[2] Theorem: a(n) does not divide a(n-1).

As $p_i < z$, $p_i^e > z$.
So any fluctuating factor, when it decreases, is > z. Hence any fluctuating factor, in any term after a certain point, is > z.
5. Suppose some stable factors exist. Define d to be the product of the stable factors. Locate 3 consecutive terms, r,s,t, after the point when degeneration has set in, and having values greater than any in that part of the sequence before degeneration set in. Locate u,v,w, later in the sequence, such that:

- u/d > max(r,s,t)
- v/d > max(r,s,t)
- w/d > max(r,s,t)

Then we could have chosen w/d instead of w, since:

- w/d < w
- it certainly is not in the sequence (is does not contain the component d, which all terms are supposed to contain)
- it satisfies the gcd

Contradiction. Therefore no stable factors exist.
6. If any increasing factor existed, it would sooner or later be > z. Locate, after that point, consecutive terms r,s,t with decreasing exponent of some fluctuating factor p_i from r to s. In particular, define the exponents of p_i in r and s to be e+x and e respectively, for positive integers e and x.
Then we could have chosen $z.p_i^{(e+x)}$ instead of t, as

- it is certainly < t
- it is not in the sequence
- it satisfies the gcd

Therefore no increasing factor exists.
7. The three types of factor were increasing, stable and fluctuating; the first two are proven to not exist, and therefore the only factors are fluctuating, with values, after a certain point, always > z.
8. Choose one prime factor, p_i. Locate r,s,t such that the exponent of p_i decreases from r to s.
In particular, define the exponents of p_i in r and s to be e+x and e respectively, for positive integers e and x.
Then we could have chosen $z.p_i^{(e+x)}$ instead of t, as:

- it is certainly < t, as the number of “other” prime factors of t is >= 1, and each other factor, also fluctuating, contributes a value > z to t.
- it is not in the sequence
- it satisfies the gcd

This is a contradiction. So the assumption, that the sequence does in fact degenerate to a series of terms that are all products of (non-trivial) powers of a single finite set of primes having k elements, where k >= 2, is false.

Thus, together with the fact that the sequence cannot degenerate to a series of terms that are all powers of a single prime p, it has been proved that, ITSII, the sequence cannot degenerate to a series of terms that are all products of (non-trivial) powers of a single finite set of primes.

Step 2. ITSII, for any prime p, there is at least one term having p as a factor.

Proof.

Proof by contradiction.
First assume the opposite: there exist some primes which divide no term in the sequence, assumed infinite. Define p as being the first such prime.

Choose now three consecutive terms r,s,t such that these and all following terms are greater than p^2.
Decompose r into its prime factors; thus r = prod(f_i^v_i) for i = 1 thru z, where z is the number of distinct prime factors of r, all f_i prime, and all $v_i > 0$.
Then for any f_i that does not divide s, we could have chosen f_ip over t, as it is certainly less that t, has not yet occurred in the sequence, and satisfies the gcd.
So all f_i divide s.
Similarly, all of the prime factors of s divide t, and so on.
As there are only a limited number of primes within the sequence (those < p), we may deduce that the sequence degenerates after a certain point, such that each term is a product of the same set of prime factors.

But this has been proved to be impossible, in Step 1.

Step 3. Define "a prime b becomes permanent" to mean that there is some integer n such that, for all i >= n, b divides a(i). Then, ITSII, there is at most 1 prime that becomes permanent.

Proof by contradiction.

By Step 2, ITSII there is no prime that is not a factor of at least one term of the sequence. By Theorem 2 on the OEIS web page, the first term having a specific prime as a factor is of the form (k^i)p for some prime k. Therefore there cannot be 2 permanent primes.

Step 4. ITSII, there is an infinite number of odd terms. In other words, the sequence does not degenerate to a series of even terms.

Proof by contradiction. Assume that the sequence degenerates to a series of even terms.

Remember from Step 2 that ITSII, no prime exists that is not a factor of some term of the sequence.

Any prime p, then, can have one of the following three behaviours:
- It disappears; there exists some n such that for all i >= n, p does not divide a(i);
- It becomes permanent; there exists some n such that for all i >= n, p does divide a(i);
- It becomes occasional; as a factor, p is present infinitely but is also absent infinitely.

Suppose then some odd prime p has its first term in the part of the sequence that has degenerated to only even terms.

1. Prove first that p, in the context of the Assumption, cannot be occasional
2. Prove next that p cannot disappear
3. Deduce then that p must be permanent, but that this leads to a contradiction.

1. If p is occasional, then there are certainly 3 consecutive terms, r,s,t (in that part of the sequence where all terms are > p) such that p divides r and p does not divide s.
Then consider p as the next term.

- Then certainly the gcd is satisfied.
- Certainly p < t.
- Certainly p is not in the sequence (it is odd).

Contradiction, therefore p cannot be occasional.

2. Suppose then that p disappears. Find then some z such that (2^z)p is not in the sequence, and is greater that any multiple of p that is present in the sequence.
Suppose t=(k^i)q, for prime k and i > 0, is the first term in the sequence (in that part where all terms are even) having prime q as a factor, for q > (2^z)p. (See Theorem 2 on the OEIS web page). Suppose the preceding terms are r,s.
Then k is 2, as (k^i)q is even.
Also, i is greater than the exponent of 2 in s, as otherwise the gcd condition would not be true.
Consider choosing (2^z)p instead of (2^i)q.

Assume first that z >= i.

- Certainly (2^z)p < q (by construction) and q < (2^i)q.
- Certainly (2^z)p is not in the sequence.
- As t = (2^i)q, we know gcd((2^i)q,r) > gcd((2^i)q,s), so as z >= i, we know that gcd((2^z)p,r) > gcd((2^z)p,s) is also true (as z >= i > exponent of 2 in s)

So we should have chosen t = (2^z)p – contradiction.

Therefore z < i.
Consider choosing (2^i)p instead of (2^i)q.

- Certainly (2^i)p < (2^i)q, as p < q.
- Certainly (2^i)p is not in the sequence, as p has already disappeared before (2^z)p, and (2^i)p > (2^z)p, as i > z.
- As p has disappeared, it does not divide r or s, and so the gcd is also true.

Contradiction, so p cannot disappear.

3. Therefore p is permanent, and the same is true of all primes > p.

But this is in contrast with Step 3.
Contradiction.
So it has been proved that ITSII, there is an infinite number of odd terms.

Step 5. ITSII, there is an infinite number of even terms.

Proof by contradiction.

Assumption. There exists a positive integer n such that a(i) is odd for all i >= n.

Using the terminology of Steps 3 and 4, primes can have one of the following three behaviours: they become occasional, they become permanent, they disappear. By Step 3, at the most one prime can become permanent.

1. Prove first, in the context of the Assumption, that no prime becomes occasional
2. Prove next that no prime becomes permanent
3. Deduce that all primes disappear, but that this leads to a contradiction.

1. If some prime p were to become occasional, and (2^i)p is not in the sequence for some i, then, in a part of the sequence where all terms are > (2^i)p, and also all terms are by now odd, choose r,s,t such that p divides r and does not divide s. Then we could have chosen (2^i)p over t, as:

- Certainly (2^i)p < t;
- Certainly (2^i)p is not in the sequence;
- It satisfies the gcd.

We must conclude that all primes (except maybe the one, if it exists, that becomes permanent), in the long term, disappear.

2. Suppose that, in the context of the assumption, indeed one prime, f, does become permanent.
Choose then some prime p so large as to be greater than f and to have its first appearance as a factor in that part of the sequence where f has already gone permanent, and all terms are even. Further, choose p so large as to be in that part of the sequence where all primes, except f, that had their first appearance before f went permanent, have since disappeared.

By Theorem 2 on the OEIS web page, the first occurrence, say a(n), of a term having p as a factor is of the form p(k^i) for some prime k, and i > 0. Therefore, k = f.
By Corollary to Theorem 2, a(n) is preceded by terms f^(i+x).b and f^(i-1).c, for:

- x >= 0
- b and c not multiples of f

By Theorem 1, s = a(n+1) is not a multiple of p.
So terms are f^(i+x)b, f^(i-1)c, (f^i)p, s.
So s = a(n+1) = (f^j)u and must satisfy gcd((f^j)u, f^(i-1)c) > gcd((f^j)u, (f^i)p).
Note that u != 1, as no simple power of f would have satisfied the gcd.

If gcd((f^j)u, f^(i-1)c) > gcd((f^j)u, (f^i)p) for

- distinct primes f and p
- u and c coprime with f and p
- i >= 1, j >= 1

then gcd(u,c) > 1.
Proof:
Suppose j >= i; then the relation is equivalent to : f^(i-1).gcd(u, c) > (f^i).gcd(u, p)
Which is equivalent to gcd(u,c) > f > 1
Suppose j < i; then the relation is equivalent to : (f^j).gcd(u, c) > (f^j).gcd(u, p)
Which is equivalent to gcd(u,c) > 1

But if this is true, a(n+1) = u would also have satisfied the gcd, has not occurred in the sequence (as is the product of primes that have only occurred in terms where f, the permanent prime, was present), and is less than (f^i)u.
This is a contradiction. So f, the permanent prime, does not exist, in the context of the Assumption at the beginning of this proof.

3. Having eliminated permanent and occasional primes, we must deduce that all primes, beyond a certain point, disappear.
Take one such prime, p, that first appears in the part of the sequence where all terms are even, and that at some time later, disappears.
Also, choose p so large that all primes that made their first appearance before the sequence degenerated to only even terms have already disappeared.
Choose the sequence r,s,t such that r is the last term having p as a factor. We must deduce that t < 2p, as otherwise we would have chosen 2p (even) over t, in that:

- It is not in the sequence;
- It would have satisfied the gcd.

Certainly t is not prime, as it would not have satisfied the gcd. Therefore t is composite with all prime factors < p.

Take q, one prime factor of t. Also for q there must be a last term, a(m), and by the same reasoning as above, a(m+2) must be composite and have a prime factor lower than q; and so on. Requiring always a lower prime leads to a contradiction when the primes run out.

Step 6. ITSII, any prime is a factor of an infinite number of terms.

Proof by contradiction. Assume that some prime p is the factor of a finite number of terms. By Step 5, p is odd.

Then there must be an integer x such that (2^x)p is not a term.
In that part of the sequence where all terms are greater than (2^x)p and no term is a multiple of p, choose consecutive terms r,s,t such that r is even and s odd (by Steps 4 and 5, such terms must exist).
Then:

- (2^x)p < t, by construction
- (2^x)p is not in the sequence
- It satisfies the gcd

Therefore we should have chosen (2^x)p instead of t.
This is a contradiction, therefore the assumption is false and it has been proved that any prime is a factor of an infinite number of terms.

Step 7. ITSII, any prime p is a term.

Step 3 states that at most one prime may become permanent.

1. Prove first that any non-permanent prime must exist as an elementary term.
2. Deduce then that no permanent prime exists, and so that, ITSII, any prime p is a term.

1. Assumption. Some non-permanent prime p is not present as an elementary term.
In that part of the sequence where all terms are greater than p, choose consecutive terms r,s,t such that r is a multiple of p and s is not (by Step 6, such terms must exist).
Then:

- $p < t$, by construction
- p is not in the sequence
- It satisfies the gcd

Therefore we should have chosen p instead of t.
Contradiction of the assumption.
Therefore any non-permanent prime p is present as an elementary term.

2. Further, as infinitely many terms are therefore prime, no permanent prime may exist, and so all primes exist as elementary terms.

Step 8. ITSII, the sequence cannot degenerate to a series of terms that all have, as a factor, at least one of a single finite set of primes. More formally, there exists no finite set of primes P with k elements, p_1 thru p_k, and integer n, such that, for all m>=n, there exists i <= k such that p_i divides a(m).

Proof. Obviously this is true, as all primes exist as a term as shown at Step 7.

Step 9. ITSII, the sequence is a permutation of the positive integers.

Proof by contradiction. Assume that some integer c is not in the sequence. We know from Step 7 that c must be composite.
In that part of the sequence where all terms are larger than c, choose some term a(m) that has a factor in common with c (such a term must exist by Step 6).
Find a(n), the first term after a(m), that has no factor in common with c (such a term must exist by Step 8).
Why didn't we choose c for a(m+1)?

- Certainly c is not in the sequence (by assumption)
- Certainly c < a(m+1) (by construction)
- Certainly c satisfies the gcd, in that it has no common factor with a(m) but does have a common factor with a(m-1)

This is a contradiction. So, ITSII, the sequence is a permutation of the positive integers.

John Mason

September 2016