The following theorems relate to the OEIS sequence A254077, which has definition: “a(n) = n if n <= 3, otherwise the smallest number not occurring earlier such that gcd(a(n),a(n-2)) > gcd(a(n),a(n-1)).”

Theorem 1. If a(n) is the first term having p (prime) as a factor, then a(n+1), if it exists, is not a multiple of p.

Theorem 2: If a(n)=cp is the first occurrence of prime p as a factor (n >3), than c has exactly one distinct prime factor. In other words, c may be expressed as k^i for some prime k, and i > 0.

Theorem 3 . If a(n) = 2p is the first term having p (prime) as a factor, then a(n-1) is odd and a(n-2) is even.

Theorem 4. If a(n) = 2p is the first term having p (prime) as a factor, then a(n+2), if it exists, is either p or 2u for some integer u such that 2u < p. (Note that it is conjectured to be always p, and observation confirms the conjecture.)

Theorem 5, generalization of Theorem 4. If a(n) = cp is the first term having p (prime) as a factor (n >3), and as a consequence c=k^i for prime k and i>0, then a(n+2), if it exists, is either p or ku for some integer u such that ku < p. (Note that it is conjectured to be always p, and observation confirms the conjecture.)

Theorem 6. If a(n) = cp is first term having p, prime, as a factor (n >3), and a(n+2)=p, then a(n+3) exists, and is not a multiple of p, and so does not terminate the sequence.

Theorem 7. If a(n) = cp is first term having p, prime, as a factor (n >3), and a(n+2)=p, then a(n+4) exists and is 2p or 3p. Also, a(n+5) exists.

Theorem 1. If a(n) is the first term having p (prime) as a factor, then a(n+1), if it exists, is not a multiple of p.

Proof.

Remember, from OEIS page A254077: "Theorem: The first occurrences of the primes as factors of terms of the sequence are in ascending order, and without gaps."

Take into account that any term of the sequence, greater than 3, must satisfy 3 conditions:

COND1: the term has never occurred before in the sequence

COND2: the term satisfies the gcd(a(n),a(n-2)) > gcd(a(n),a(n-1))

COND3: the term is the lowest possible value to satisfy COND1 and COND2

The proof of the Theorem is based on the assumption of its opposite, showing that this assumption leads to a contradiction.

Assumption 1: Assume a(n+1) exists and is a multiple of p

Step 1: Show that the terms a(n-2) thru a(n+1) may be expressed as r,fcx,tp,fcyp for prime f.

> Denote the 4 terms from a(n-2) thru a(n+1) thus: r,s,tp,up.
> Then by sequence definition gcd(up,s) > gcd(up,tp).
> Therefore gcd(u,s)>p.
> Therefore u > p, and s > p.
> Define e = gcd(u,s), with e > p
> Substitute s with ex and u with ey, where x and y are coprime.
> Terms become r,ex,tp,eyp.
> e > p and ex precedes tp. Therefore, from previous theorem on OEIS page, e is composite, and has prime factors all < p.
> Define f as least prime factor of e. Then f < p.
> Substitute e with fc.
> Terms become r,fcx,tp,fcyp.

Terminology: Define q as next prime after p.

Step 2: Show that, within Assumption 1, f and t are not coprime. This is proved by assuming the opposite.

Assumption 2 (within Assumption 1): Assume f and t are coprime.

> **Step 2.1**: Show that the terms a(n-2) thru a(n+1) may be expressed as r,fcx,tp,fcp
>
> > We know q < 2p by Bertrand's Postulate.
> > So fq < 2p^2.
> > Could we have chosen fq over fcyp as term a(n+1)?
> > Consider COND2.
> > gcd(fq,fcx)=f.gcd(q,cx)=f (as q is certainly not a factor of terms thru a(n)).
> > gcd(fq,tp)=1.
> > Therefore gcd(fq,fcx)>gcd(fq,tp).
> > COND1: fq is certainly not already in the sequence.
> > Since we chose fcyp over fq, we must deduce fq>fcyp (COND3).
> > Therefore 2pp>fq>fcyp (since 2p > q and p > f).

Hence 2pp > fcyp.
Hence fcy < 2p.
Hence ey < 2p.
But e > p, so y=1.
Terms become r,fcx,tp,fcp.

Step 2.2: Show that c = 1

Assumption 3 within Assumption 2: Assume c > 1.

Could we have chosen fq over fcp as term a(n+1)?
COND1: fq is certainly not already in the sequence.
COND3: Certainly fq < fcp (again, by Bertrand's Postulate).
Is gcd(fq,fcx) > gcd(fq,tp) ? (COND2)
This is equivalent to: is f > 1 (remembering Assumption 2)? Which is obviously true (f is prime).
Therefore we should have chosen fq over fcp.
Therefore Assumption 3 is false and c = 1.

Step 2.3: Show then that Assumption 2 leads to a contradiction

Terms become r,fx,tp,fp where f and t are coprime (Assumption 2).
gcd(fp,fx)>gcd(tp,fp) by sequence definition.
So f > p, which is a contradiction.
So Assumption 2 is false, f & t are not coprime, and so f (prime) divides t.

Step 3: Show that c > p

Substitute t with fm.
Terms become r,fcx,fmp,fcyp.
Therefore by sequence definition : gcd(fcyp,fcx) > gcd(fcyp,fmp).
So fc.gcd(yp,x) > fp.gcd(cy,m).
So fc > fp.gcd(cy,m) (remembering x and y are coprime, and p does not divide x).
So c > p.gcd(cy,m).
So c > p.

Step 4: Show that f & cy are not coprime

Assumption 4 within Assumption 1: Assume f & cy are coprime

Step 4.1: Prove y = 1 (within Assumption 4)

Assumption 5 within Assumption 4: Assume y > 1

Could we have chosen fcq over fcyp?
COND1: fcq is certainly not already in the sequence.
COND3: fcq < fcyp (as q < 2p <= yp).
Therefore gcd(fcq,fcx) <= gcd(fcq,fmp) , because otherwise we would have chosen fcq, or something less.
So fc <= f.gcd(c,m).
So m is a multiple of c; suppose m=zc.
Terms become r,fcx,fzcp,fcyp.

So gcd(fcyp,fcx) > gcd(fzcp,fcyp).
So fc > fcp.gcd(z,y).
So fc > fcp which is impossible, so Assumption 5 is false and y = 1.

Step 4.2: Show that Assumption 4 leads to a contradiction

Sequence (within Assumption 4) is : r,fcx,fmp,fcp with f & c coprime.
Could we have chosen cq over fcp?
COND1: cq is certainly not already in the sequence.
COND3: Certainly cq < fcp (as q < 2p <= fp).
So gcd(cq,fcx) <= gcd(cq,fmp) , because otherwise we would have chosen cq, or something less.
So c <= gcd(c,fm), but f & c are coprime (by Assumption 4: f & cy are coprime, and also y = 1).
So c <= gcd(c,m) which is possible only if m is a multiple of c.
Terms become r,fcx,fzcp,fcp.
But fcp divides fzcp which contradicts proof in OEIS A254077 web page : a(n) does not divide a(n-1).
So Assumption 4 is false and f and cy are not coprime.

Recap:

Sequence is r,fcx,fmp,fcyp.
f,p are prime.
f and cy are not coprime.
x and y are coprime.

Step 5: Show that f does not divide y

Assumption 6 within Assumption 1: assume f divides y, and y=df

Remember x & y are coprime, so d and f are coprime with x.
Terms become r,fcx,fmp,fcdfp.
Could we have chosen fcq over fcdfp?
COND1: fcq is certainly not already in the sequence.
COND3: Certainly fcq < fcdfp.
Therefore, as we did choose fcdfp over fcq, gcd(fcq,fcx) <= gcd(fcq,fmp).
So fc <= f.gcd(m,c).
So m is a multiple of c; suppose m=zc.
Terms become r,fcx,fzcp,fcdfp.
So gcd(fcdfp, fcx) > gcd(fcdfp, fzcp).
So fc > fcp – contradiction.
So Assumption 6 is false and f does not divide y; since f and cy are not coprime, we know that c is a multiple of f; say c = gf.

Step 6: Prove y = 1

Terms are r,fgfx,fmp,fgfyp

Assumption 7 within Assumption 1: Assume y > 1

Could we have chosen fgfq over fgfyp?

Certainly fgfq < fgfyp (COND3)
COND1: fgfq is certainly not already in the sequence.
So gcd(fgfq,fgfx) <= gcd(fgfq,fmp) otherwise we would have chosen fgfq or less
So fgf <= f.gcd(gf,m).
So m is a multiple of gf; suppose m=hgf.
Terms become r,fgfx,fhgfp,fgfyp.
So gcd(fgfyp, fgfx) > gcd(fgfyp, fhgfp).
So fgf > fgfp.gcd(y,h) which is impossible.
So Assumption 7 is false and y = 1.

Step 7: Show that Assumption 1 leads to a contradiction

Terms become r,fgfx,fmp,fgfp.
Could we have chosen fgq over fgfp?
Certainly fgq < fgfp (COND3).
COND1: fgq is certainly not already in the sequence.
So gcd(fgq,fgfx) <= gcd(fgq,fmp) otherwise we would have chosen fgq or less.
So fg <= f.gcd(g,m).
So m is a multiple of g; say m=gj.
Terms become r,fgfx,fgjp,fgfp.
So gcd(fgfp, fgfx) > gcd(fgfp, fgjp).
So fgf > fgp.gcd(f,j).
Which is impossible, as f < p.
Contradiction of Assumption 1.

Therefore theorem is proved

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Theorem 2: If a(n)=cp is the first occurrence of prime p as a factor (n >3), than c has exactly one distinct prime factor. In other words, c may be expressed as k^i for some prime k, and i > 0.

Proof

Express terms a(n-2), a(n-1), a(n) as r,s,cp (and remember that by proof on OEIS page, c > 1)

Prove by contradiction, by first assuming the opposite, that for some a(n)=cp, the first occurrence of prime p as a factor (n >3), c has for than one distinct prime factor

Therefore, express c as product(k_i^v_i) for i=1 thru z, for prime k_i and integer v_i > 0, where z is the number of distinct prime factors of c, and z >= 2

Prove first that, according to this assumption, for each i, that k_i^v_i divides r, and the maximum power of k_i that divides s is precisely k_i^(v_i – 1)

Remember that gcd(cp,r) > gcd(cp,s)
So gcd(product(k_i^v_i) for i=1 thru z,r) > gcd(product(k_i^v_i) for i=1 thru z,s), and c is the lowest value to satisfy this condition.
Therefore, if, for some i, k_i^v_i does not divide r, a lesser value of c, having a smaller power of k_i would have satisfied the gcd.
Also, if the maximum power of k_i that divides s were less than k_i^(v_i – 1), then a lesser value of c, having a smaller power of k_i would have satisfied the gcd. If k_i^v_i were a factor of s, then a lesser value of c, not having k_i as a factor would have satisfied the gcd.

We have therefore proved that for each i, that k_i^v_i divides r, and the maximum power of k_i that divides s is precisely k_i^(v_i – 1)
So the sequence may be expressed as tc,uy,cp where

- y is the product of the prime factors of c reduced by one power
- u and c are coprime

Could we have chosen a(n)=(k_1^v_1)p ?
Certainly it is not in the sequence.
Certainly it is less than c.
Is the gcd true?
Consider if gcd((k_1^v_1)p,tc) > gcd((k_1^v_1)p,uy)
This is equivalent to k_1^v_1 > k_1^(v_1 – 1), which is true.
So (k_1^v_1)p would have been a better candidate for a(n).
Therefore assumption is false and Theorem is proved.

Corollary. If a(n)=cp is the first occurrence of prime p as a factor (n >3), and as a consequence c= k^i for some prime k, and i > 0, then k^i divides a(n-2) and k^(i-1) is the maximum power of k that divides a(n-1).

Proof

Express terms a(n-2), a(n-1), a(n) as r,s,(k^i)p
Then if k^i does not divide r, then some lower value of x would have done.
Then if k^(i-1) does not divide s, then some lower value of x would have done.
If k^i divided s, then the gcd would not be true.

Theorem 3 . If a(n) = 2p is the first term having p (prime) as a factor, then a(n-1) is odd and a(n-2) is even.

Proof.

Define terms a(n-2), a(n-1), a(n) as r,s,2p for p, prime, occurring for the first time in the sequence.
So gcd(2p,r) > gcd(2p,s)
So gcd(2,r) > gcd(2,s)
So gcd(2,r)=2 and gcd(2,s)=1

Theorem 4. If a(n) = 2p is the first term having p (prime) as a factor, then a(n+2), if it exists, is either p or 2u for some integer u such that 2u < p. (Note that it is conjectured to be always p, and observation confirms the conjecture.)

Proof.

The value p satisfies the gcd, is coprime (by Theorem 1) with a(n+1), and is not yet in the sequence.
Therefore a(n+2) is either p or some lower value.
Assume a(n+2)=z < p, and sequence is 2p,y,z
Then gcd(z,2p) > gcd(z,y) >= 1
Therefore z=2u for integer u, and 2u < p

Theorem 5, generalization of Theorem 4. If a(n) = cp is the first term having p (prime) as a factor (n >3), and as a consequence c=k^i for prime k and i>0, then a(n+2), if it exists, is either p or ku for some integer u such that ku < p. (Note that it is conjectured to be always p, and observation confirms the conjecture.)

Proof.

The value p satisfies the gcd, is coprime (by Theorem 1) with a(n+1), and is not yet in the sequence.
Therefore a(n+2) is either p or some lower value.
Assume a(n+2)=z < p, and sequence is cp,y,z
By previous proof, c=k^i, i > 0
So sequence is (k^i)p,y,z
Then gcd(z,cp) > gcd(z,y) >= 1
Therefore z=ku for integer u, and ku < p

Theorem 6. If a(n) = cp is first term having p, prime, as a factor (n >3), and a(n+2)=p, then a(n+3) exists, and is not a multiple of p, and so does not terminate the sequence.

Proof

Certainly a(n+3) exists, as a(n+1) does not divide a(n+2) (see proof on OEIS page).
Prove now that a(n+3) is not a multiple of p.

Proof by contradiction.

Assumption 1. Suppose sequence is cp,s,p,zp.
So gcd(zp,s)>gcd(zp,p)
So gcd(z,s)>p
So z > p and s > p

Assumption 2. Suppose that z is prime.
As gcd(z,s) > p, so z, prime, divides s.
Therefore a(n+1)=s is the first occurrence of z as a prime.
Therefore a(n+3), if it exists, is <= z (by Theorem 5), which is a contradiction.

Therefore Assumption 2 is false and z is composite.

Assumption 3. Suppose that s is prime.
As s > p, and primes appear as factors in order (see proof on OEIS page), we know that a(n+1) is the first occurrence of s as a factor. But the first occurrence of a prime (>3) is never the prime itself (see proof on OEIS page), so s being prime leads to a contradiction.

So Assumption 3 is false and s is composite.

Define f = lowest prime factor of gcd(s,z), remembering that gcd(s,z) > p and so certainly has prime factors.
Rewrite sequence cp,fg,p,fhp. As z is composite, we know h >= 2
Define q = next prime after p
Could we have chosen a(n+3)=fq?
Consider if gcd(fq,fg)>gcd(fq,p).
We know f, p and q are prime, so the relation is equivalent to f.gcd(q,g) > 1, which is true.
We know fq < fhp, as q < 2p (by Bertrand's Postulate)
Therefore, as we chose fhp over fq, this latter must be already in the sequence.
The first available place is a(n+1), which is therefore the first occurrence of q as a prime.
Therefore a(n+3) <= q (by Theorem 5)
But fhp > q : contradiction.

So Assumption 1 is false, a(n+3) is not a multiple of p, and so does not terminate the sequence.

So Theorem is proved.

Theorem 7. If a(n) = cp is first term having p, prime, as a factor (n >3), and a(n+2)=p, then a(n+4) exists and is 2p or 3p. Also, a(n+5) exists.

Proof

By Theorem 6, a(n+4) exists.
Therefore terms are cp,s,p,x,y
We know that p does not divide x, by Theorem 6
So gcd(y,p)>gcd(y,x)>=1
So gcd(y,p)>1
So y is a multiple of p, and is not equal to p.
Further, as a(n) was the first occurrence of p as a factor, if c=2, then y=3p; otherwise, y=2p.

Also, as x is not 2, is not 3, and is not p, it does not divide a(n+4), and so a(n+5) exists.

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