Fractional iteration of a series inversion operator

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We consider an operator $\mathcal{I}$ on formal power series, closely related to the series reversion operator, and show how to define complex iterates $\mathcal{I}^\alpha$ of $\mathcal{I}$. The power series expansion of $\mathcal{I}^\alpha(f(x))$ is found. Examples include the generalized binomial and generalized exponential series of Lambert.

We begin by recalling some facts about formal power series [4].

Let $g(x)$ be a formal power series with complex coefficients of the form

$$g(x) \quad = \quad x + g_2x^2 + g_3x^3 + \cdots .$$

There is a unique formal power series $h(x)$, called the compositional inverse of $g(x)$, such that

$$h(g(x)) = g(h(x)) = x.$$

The process of finding the coefficients of the series $h(x)$ from those of $g(x)$ is called series reversion or series inversion. We denote $h(x)$ by $\mathrm{Rev}(g(x))$, where Rev is the series reversion operator.

Let

$$f(x) \quad = \quad 1 + f_1x + f_2x^2 + f_3x^3 + \cdots \tag{1}$$

be a formal power series with constant term 1. We can raise a power series of this type to an arbitrary complex power α. Indeed, if we write f in the form

$$f(x) \quad = \quad 1 + S(x),$$

with $S(x) = \sum_{n\ge1} f_nx^n$, then we have, by virtue of the generalized binomial theorem,

$$f^\alpha(x) \quad = \quad 1 + \sum_{n\ge1}\binom{\alpha}{n}S^n(x),$$

Collecting together the powers of x we obtain

$$f^\alpha(x) \quad = \quad \sum_{n\ge0} p_n(\alpha)x^n,$$

a well-defined formal power series, where $p_n(t)$ is a sequence of convolution polynomials [3].

Definition 1. We define a series inversion operator $\mathcal{I}$, acting on power series $f(x)$ of the form (1), by

$$\mathcal{I}(f(x)) = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{f(x)}\right). \tag{2}$$

The Lagrange inversion formula gives the coefficents in the power series expansion of $\mathcal{I}(f(x))$ as

$$[x^n]\,\mathcal{I}(f(x)) = \frac{1}{n+1}\,[x]^n\,f(x)^{n+1}.$$

The power series $\mathcal{I}(f(x))$ has the same form as $f(x)$ with constant term 1. Thia allows us to iterate the inversion operator $\mathcal{I}$.

Example 1. Let $f(x) = 1/(1-x) = 1 + x + x^2 + x^3 + \cdots$, the generating function for the all 1's sequence, A000012. Then

$$\begin{aligned}\mathcal{I}(f(x)) &= \frac{1}{x}\mathrm{Rev}\,(x(1-x)) \\ &= \frac{1-\sqrt{1-4x}}{2x}\end{aligned}$$

is the generating function for the Catalan numbers, A000108. One (of the many) combinatorial interpretations of the Catalan numbers is as the number of full binary trees with n internal vertices. By a slight abuse of notation we display this relationship between the two sequences diagramatically as

$$A000012 \xrightarrow{\mathcal{I}} A000108$$

Iteration of the operator $\mathcal{I}$ produces the following diagram of OEIS sequences.

$$A000108 \xrightarrow{\mathcal{I}} A001764 \xrightarrow{\mathcal{I}} A002293 \xrightarrow{\mathcal{I}} A002294 \xrightarrow{\mathcal{I}} A002295 \xrightarrow{\mathcal{I}} A002296 \xrightarrow{\mathcal{I}} \cdots$$

Combinatorially, these sequences enumerate k-ary trees for $k = 2, 3, 4, ...$

The opertor $\mathcal{I}$ is invertible with inverse denoted $\mathcal{I}^{-1}$. It follows from (2) that

$$\mathcal{I}^{-1}(f(x)) = \frac{x}{\mathrm{Rev}(xf(x))},$$

which we can write in the more suggestive form

$$\mathcal{I}^{-1}(f(x)) = \left(\frac{1}{x}\mathrm{Rev}\left(\frac{x}{f^{-1}(x)}\right)\right)^{-1}, \tag{3}$$

where the exponent -1 occurring on the right-hand side means reciprocation of a function.

Definition 2. Let $\alpha \in \mathbb{C}$. Let $f(x)$ be a power series of the form (1). We define $\mathcal{I}^{\alpha}$, the fractional inversion operator of order α, by putting $\mathcal{I}^{0}(f(x)) = f(x)$ and setting

$$\begin{aligned} \mathcal{I}^{\alpha}\left(f(x)\right) &= \left(\mathcal{I}(f^{\alpha}(x))\right)^{\frac{1}{\alpha}} \\ &= \left(\frac{1}{x}\mathrm{Rev}\left(\frac{x}{f^{\alpha}(x)}\right)\right)^{\frac{1}{\alpha}} \text{ for } \alpha \neq 0. \end{aligned} \tag{4}$$

Clearly, this definition agrees with (3) when $\alpha = -1$, and we also have $\mathcal{I}^{1} = \mathcal{I}$.

We have used the same misnomer here as in fractional calculus by referring to the operator $\mathcal{I}^{\alpha}$ as a fractional inversion operator, even when α is not a rational number.

In order for the definition of fractional inversion to be useful we would like the operator $\mathcal{I}^{n}$ to equal the n-fold iterate of $\mathcal{I}$ when n is a positive integer, and equal the n-fold iterate of $\mathcal{I}^{-1}$ when n is a negative integer. This is true and is an immediate consequence of the following result.

Theorem 1. *Let α, β be complex numbers. Then*

$$\mathcal{I}^{\alpha} \circ \mathcal{I}^{\beta} = \mathcal{I}^{\alpha+\beta}.$$

Proof. The result is obviously true when either $\alpha = 0$ or $\beta = 0$ and is straightforward to verify when $\alpha + \beta = 0$. From now on we assume that α, β and $\alpha + \beta$ are all non-zero.

Let $f(x) = 1 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$ be a formal power series and define

$$g(x) = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{f^{\beta}(x)}\right). \tag{5}$$

From the definition (4) of the fractional inversion operator we have

$$\begin{aligned} \mathcal{I}^{\beta}(f(x)) &= \left(\frac{1}{x}\mathrm{Rev}\left(\frac{x}{f^{\beta}(x)}\right)\right)^{\frac{1}{\beta}} \\ &= g^{\frac{1}{\beta}}(x). \end{aligned}$$

Hence

$$\begin{aligned}\mathcal{I}^\alpha\left(\mathcal{I}^\beta(f(x)\right) &= \mathcal{I}^\alpha\left(g^{\frac{1}{\beta}}(x)\right)\\ &= \left(\frac{1}{x}\mathrm{Rev}\left(\frac{x}{g^{\frac{\alpha}{\beta}}(x)}\right)\right)^{\frac{1}{\alpha}}. \qquad (6)\end{aligned}$$

If we put

$$h(x) = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{g^{\frac{\alpha}{\beta}}(x)}\right) \qquad (7)$$

then (6) reads

$$\mathcal{I}^\alpha\left(\mathcal{I}^\beta(f(x)\right) = h^{\frac{1}{\alpha}}(x). \qquad (8)$$

Again, by the definition of the fractional inversion operator, we have

$$\mathcal{I}^{\alpha+\beta}(f(x)) = \left(\frac{1}{x}\mathrm{Rev}\left(\frac{x}{f^{\alpha+\beta}(x)}\right)\right)^{\frac{1}{\alpha+\beta}}. \qquad (9)$$

If we put

$$H(x) = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{f^{\alpha+\beta}(x)}\right) \qquad (10)$$

then (9) becomes

$$\mathcal{I}^{\alpha+\beta}(f(x)) = H(x)^{\frac{1}{\alpha+\beta}}. \qquad (11)$$

Comparing (8) and (11), the proof that $\mathcal{I}^\alpha \circ \mathcal{I}^\beta = \mathcal{I}^{\alpha+\beta}$ will be established if we can show

$$h(x)^{\frac{1}{\alpha}} = H(x)^{\frac{1}{\alpha+\beta}}. \qquad (12)$$

Now from (5), the power series $xg(x)$ is the inverse of $x/f^\beta(x)$, which leads to the functional equation

$$g\left(\frac{x}{f^\beta(x)}\right) = f^\beta(x)$$

or equivalently

$$g^{\frac{1}{\beta}}\left(\frac{x}{f^\beta(x)}\right) = f(x). \qquad (13)$$

Similarly, from (7) we find that the power series $h(x)$ satisfies the functional equation

$$h^{\frac{1}{\alpha}}\left(\frac{x}{g^{\frac{\alpha}{\beta}}(x)}\right) = g^{\frac{1}{\beta}}(x) \qquad (14)$$

while from (10), the power series $H(x)$ satisfies the functional equation

$$H^{\frac{1}{\alpha+\beta}}\left(\frac{x}{f^{\alpha+\beta}(x)}\right) \quad = \quad f(x). \tag{15}$$

In (14), replace x with $\frac{x}{f^{\beta}(x)}$ and then use (13) to find that h satisfies

$$h^{\frac{1}{\alpha}}\left(\frac{x}{f^{\alpha+\beta}(x)}\right) \quad = \quad f(x). \tag{16}$$

Thus from (15) and (16) we have

$$H^{\frac{1}{\alpha+\beta}}(x) \quad = \quad f\left(\mathrm{Rev}\left(\frac{x}{f^{\alpha+\beta}(x)}\right)\right) = h^{\frac{1}{\alpha}}(x),$$

proving (12) and completing the proof of the Theorem. □

Corollary 1. *Let α, β be complex numbers. The operators $\mathcal{I}^{\alpha}$ and $\mathcal{I}^{\beta}$ commute*

$$\mathcal{I}^{\alpha} \circ \mathcal{I}^{\beta} \quad = \quad \mathcal{I}^{\beta} \circ \mathcal{I}^{\alpha}.$$

Corollary 2. *For a positive integer n, the operator $\mathcal{I}^{n}$ equals the n-fold iterate of $\mathcal{I}$ and the operator $\mathcal{I}^{-n}$ equals the n-fold iterate of $\mathcal{I}^{-1}$.*

$$\mathcal{I}^{n} \quad = \quad \mathcal{I} \circ \ldots \circ \mathcal{I} \quad (n \text{ factors})$$

$$\mathcal{I}^{-n} \quad = \quad \mathcal{I}^{-1} \circ \ldots \circ \mathcal{I}^{-1} \quad (n \text{ factors})$$

Exercise 1. Let R denote the reciprocation operator: $R : f(x) \to 1/f(x)$. Show $R \circ \mathcal{I}^{\alpha}$ and $\mathcal{I}^{\alpha} \circ R$ are both idempotent operators.

Exercise 2. Show Theorem 1 has the following generalization: Let $f(x) = 1 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$ *be a formal power series. Let α, β, r be complex numbers. Then*

$$\mathcal{I}^{\alpha}\left(\left(\mathcal{I}^{\beta}(f)\right)^{r}\right) \quad = \quad \left(\mathcal{I}^{r\alpha+\beta}(f)\right)^{r}.$$

Corollary 2 reveals the surprising property of the inversion operator $\mathcal{I}$ that its n-th iterate can be calculated from a single application of $\mathcal{I}$. The next result gives an alternative way of computing $\mathcal{I}^{n}(f(x))$ that also requires only a single application of $\mathcal{I}$.

Theorem 2. Let $f(x) = 1 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots$ be a formal power series. Let n be a positive integer. Then

(i)

$$\mathcal{I}^n(f(x)) = [\mathcal{I}(f(x^n))]_{x \to x^{\frac{1}{n}}}$$

(ii)

$$\mathcal{I}^{-n}(f(x)) = \left[\mathcal{I}^{-1}(f(x^n))\right]_{x \to x^{\frac{1}{n}}}$$

Proof. We prove (i), the proof of (ii) being exactly similar.

The claim is that $\mathcal{I}^n(f(x))$ is obtained from $\mathcal{I}(f(x^n))$ (a power series in x^n) on replacing x with $x^{\frac{1}{n}}$.

Let $G(x) = \mathcal{I}^n(f(x))$ and $H(x) = \mathcal{I}(f(x^n))$. We show $G(x) = H\left(x^{\frac{1}{n}}\right)$.

By definition (4) of the fractional inversion operator we have

$$G(x) = \left(\frac{1}{x}\text{Rev}\left(\frac{x}{f^n(x)}\right)\right)^{\frac{1}{n}}. \tag{17}$$

It follows that $xG^n(x)$ is the series reversion of $x/f^n(x)$, and consequently

$$\frac{xG^n(x)}{f^n\left(xG^n(x)\right)} = x.$$

Thus $G(x)$ satisfies the functional equation

$$G(x) = f\left(xG^n(x)\right). \tag{18}$$

Moreover, this functional equation determines $G(x)$ in terms of $f(x)$ because, starting from (18), we can reverse the above steps to find $G(x)$ is given by (17).

It follows from $H(x) = \mathcal{I}(f(x^n))$ that $xH(x)$ is the series reversion of $x/f(x^n)$, and hence

$$\frac{xH(x)}{f\left(x^n H^n(x)\right)} = x$$

giving

$$H(x) = f\left(x^n H^n(x)\right).$$

Thus

$$H\left(x^{\frac{1}{n}}\right) = f\left(xH^n\left(x^{\frac{1}{n}}\right)\right). \tag{19}$$

Comparing (18) and (19) we see that $G(x)$ and $H(x^{\frac{1}{n}})$ satisfy the same functional equation, which, as we noted above, has a unique solution in terms of $f(x)$. Consequently $G(x) = H(x^{\frac{1}{n}})$. □

We turn our attention to the series expansion of $\mathcal{I}^t\left(f(x)\right)$ and its powers. We shall need the following version of the Lagrange-Bürmann formula for formal power series (see [1, Theorem 1.2.4] or [5]): If
$f(x) = 1 + f_1x + f_2x^2 + f_3x^3 + \cdots$ and $H(x) = h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots$ are formal power series and $G(x) = \text{Rev}\left(\frac{x}{f(x)}\right)$ then

$$[x^n]H(G(x)) = \frac{1}{n}\left[x^{n-1}\right]H^{'}(x)f(x)^n, \text{ for } n, k > 0.$$

Theorem 3. Let

$$f(x) = 1 + f_1x + f_2x^2 + f_3x^3 + \cdots$$

be a formal power series and let

$$f(x)^t = \sum_{n\geq 0} p_n(t)x^n,$$

where $p_n(t)$ is a family of convolution polynomials. Then for $r \in \mathbb{C}$,

(i)

$$\left(\mathcal{I}^t(f(x))\right)^r = \sum_{n\geq 0} \frac{r}{nt+r}p_n(nt+r)x^n$$

(The denominator $nt + r$ cancels with the corresponding factor in the numerator polynomial $p_n(nt + r)$, so no problem arises if $nt + r$ is zero.)

(ii)

$$\log\left(\mathcal{I}^t(f(x))\right) = \sum_{n\geq 1} \frac{1}{nt}p_n(nt)x^n.$$

Proof. (i) The result is clearly true if $t = 0$. Assume now t is nonzero. By definition

$$\mathcal{I}^t(f(x)) = \left(\frac{1}{x}\text{Rev}\left(\frac{x}{f(x)^t}\right)\right)^{\frac{1}{t}}.$$

Let $g(x) = (\mathcal{I}^t(f(x)))^t$ so that

$$xg(x) = \text{Rev}\left(\frac{x}{f(x)^t}\right).$$

Now apply the Lagrange-Bürmann formula with $H(x) = x^k, k > 0$ to obtain

$$[x^n]\left(xg(x)\right)^k = \frac{1}{n}[x^{n-1}]kx^{k-1}f(x)^{nt}$$

or equivalently

$$[x^{n-k}]\,(g(x))^k \quad = \quad \frac{k}{n}[x^{n-k}]f(x)^{nt}.$$

Replace n with $n+k$ to get

$$\begin{aligned}[x^{n}]\,(g(x))^k &= \frac{k}{n+k}[x^{n}]f(x)^{(n+k)t}\\ &= \frac{k}{n+k}p_n((n+k)t).\end{aligned}$$

Thus we have the series expansion

$$g(x)^k = (\mathcal{I}^t(f(x)))^{kt} \quad = \quad \sum_{n\geq 0}\frac{k}{n+k}p_n((n+k)t)x^n. \tag{20}$$

Equation (20) has been derived on the assumption that k is a positive integer but also holds for arbitrary complex k, since when k is a complex number the coefficients of $g(x)^k$ are polynomials in k that equal the polynomials $\frac{k}{n+k}p_n((n+k)t)$ for infinitely many values of k and so must be identically equal polynomials.

Let $r \in \mathbb{C}$ and set $k = r/t$ in (20) to find

$$(\mathcal{I}^t(f(x)))^r \quad = \quad \sum_{n\geq 0}\frac{r}{nt+r}p_n(nt+r)x^n. \tag{21}$$

In particular,

$$\mathcal{I}^t(f(x)) \quad = \quad \sum_{n\geq 0}\frac{1}{nt+1}p_n(nt+1)x^n$$

(ii) It follows from (21) that

$$\begin{aligned}\log\left(\mathcal{I}^t(f(x))\right) &= \lim_{r\to 0}\frac{\mathcal{I}^t(f(x))^r-1}{r}\\ &= \sum_{n\geq 1}\frac{1}{nt}p_n(nt)x^n. \quad \square\end{aligned}$$

One consequence of Theorem 3 (i) is that if $p_n(t)$ is a family of convolution polynomials then $q_n(r) := \frac{r}{nt+r}p_n(nt+r)$, regarded as polynomials in the variable r, is another family of convolution polynomials [2, pp. 15-16].

Theorem 3 coupled with Exercise 2 allows us to write down the effect of the fractional inversion operator $\mathcal{I}^s$ applied to powers of $\mathcal{I}^t(f(x))$:

$$\begin{aligned} \mathcal{I}^s\left(\left(\mathcal{I}^t(f(x))\right)^r\right) &= \left(\mathcal{I}^{rs+t}(f(x))\right)^r \\ &= \sum_{n\geq 0} \frac{r}{n(rs+t)+r} p_n(n(rs+t)+r)x^n. \end{aligned}$$

We conclude by looking at two classical families of series which may be defined using the fractional inversion operator.

Example 2. Take $f(x) = 1 + x$. Let $\mathcal{B}_t(x)$ denote the power series $\mathcal{I}^t(f(x)) = \mathcal{I}^t(1+x)$. Now $f(x)^t = \sum_{n\geq 0}\binom{t}{n}x^n$, so in this case $p_n(t) = \binom{t}{n}$ is a falling factorial polynomial. Theorem 3 gives the expansion

$$\mathcal{B}_t(x) = \sum_{n\geq 0} \frac{1}{nt+1}\binom{nt+1}{n}x^n \tag{22}$$

with powers given by

$$\mathcal{B}_t(x)^r = \sum_{n\geq 0} \frac{r}{nt+r}\binom{nt+r}{n}x^n. \tag{23}$$

Also

$$\log\left(\mathcal{B}_t(x)\right) = \sum_{n\geq 1} \frac{1}{nt}\binom{nt}{n}x^n.$$

This latter series (with t replaced by $1+t$) is the exponential generating function for A056856.

The series $\mathcal{B}_t(x)$ are called generalized binomial series and have a long history dating back to Lambert. See [2, Section 5.4 and Section 7.5] and A251592. For an alternative approach to proving (23), avoiding the Lagrange-Bürmann formula and using only basic calculus, see 'The power series for the inverse function of y(1-y)^t' by N. D. Elkies, available online at http://www.math.harvard.edu/~elkies/Misc/catalan.pdf.

Example 3. Take$f(x) = e^x$. Let $\mathcal{E}_t(x)$ denote the power series $\mathcal{I}^t(f(x)) = \mathcal{I}^t(e^x)$. We have $f(x)^t = e^{xt} = \sum_{n\geq 0} t^n\frac{x^n}{n!}$, so in this case $p_n(t) = t^n$ is a monomial. Theorem 3 gives the expansion

$$\mathcal{E}_t(x) = \sum_{n\geq 0}(nt+1)^{n-1}\frac{x^n}{n!}. \tag{24}$$

In the particular case $t = 0$ we have $\mathcal{E}_0(x) = e^x$. Graham et al. [2, Section 5.4] call $\mathcal{E}_t(x)$ a generalized exponential series. See A139526.

The powers of the generalized exponential series may also be written down

$$\mathcal{E}_t(x)^r = \sum_{n\geq 0} r(nt+r)^{n-1}\frac{x^n}{n!}\,. \tag{25}$$

Furthermore,

$$\log\left(\mathcal{E}_t(x)\right) = \sum_{n\geq 1} (nt)^{n-1}\frac{x^n}{n!}\,. \tag{26}$$

In particular,

$$\log\left(\mathcal{E}_1(x)\right) = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!} \tag{27}$$

is Euler's tree function. See A000169.

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