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A Note on the Area Table A249869 for Primitive Pythagorean Triangles

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Abstract

For the table A249869 of areas of primitive Pythagorean triangles a row number N is determined such that all areas not exceeding a given A_{top} are reached by taking into account the rows numbered 2 to N. This N will in general not be minimal. We give the list of all areas for primitive Pythagorean triangles up to 10^6 with multiplicities and their squarefree parts.

1. Introduction

The number triangle A249869 $T(n, m)$, $n = 2, 3, ...$ and $m = 1, 2, ..., n-1$, collects the areas A (in some length square units) of primitive integer Pythagorean triangles (primitive PTs) given by a triple (x, y, z) with positive integers x, y, z, even y, and $\gcd(x, y, z) = 1$. The problem is to find the row number N such that all areas up to and including a given area A_{top} are among the entries of the rows $n = 2, ..., N$. This is necessary to find from this triangle the list of all areas of primitive PTs with areas $A \leq A_{top}$ given in A024406.

The famous problem of the so-called congruent numbers a (congruent not in the sense used in the context of residue classes) [2] [1] is related to areas of rational PTs (X, Y, Z). Because each such triangle with area a corresponds to an integer PT after multiplying the sides with the lcm (least common multiple) of the denominators of these three rationals (in lowest terms) one has to search for the positive integer areas $A = lcm^2\, a$ of integer PTs. Conversely, if a primitive integer PT has an area A which is not squarefree, *i.e.*, $A = s^2\, a$, $s > 1$, with squarefree a, then there is a corresponding rational PT with area a. Such squarefree integer areas a are called congruent numbers (sometimes called primitive congruent numbers). For example, the smallest area of an integer PT (necessarily primitive) is $A = 6$ from the triple $(3, 4, 5)$ (see A024406), but this is not the smallest congruent number which is 5 (see A006991), coming from the integer PT with non-squarefree area $A = 180 = 6^2 \cdot 5$, *i.e.*, $(9, 40, 41)$, which corresponds to the rational PT $\left(\frac{3}{2}, \frac{20}{3}, \frac{41}{6}\right)$. Going up to row $N = N(A_{top})$ in the table will find from the squarefree parts also different PTs leading to the same congruent number. For example the congruent number 6 is also obtained from a rational PT coming from the non-squarefree area $T(25,\, 24) = 29400 = 70^2 \cdot 6$, for the triple $(49, 1200, 1201)$, leading to the rational PT $\left(\frac{7}{10}, \frac{120}{7}, \frac{1201}{70}\right)$.

To find all congruent numbers $a \leq a_{top}$ from the primitive PT areas of table A249869 is impossible because one never knows up to which entries A_{top} ane has to go. For example, to find the congruent number 157, corresponding to the rational *Zagier* PT (see [2]. p. 5) one has to go up to $A = s^2\, 157$ with the 46 digit number s given as denominator of the hypothenuse, resulting in a 95 digit number for A. Therefore, a knowledge of the primitive PT area table (up to any finite row number) does not provide a criterion for congruent numbers. However, one can find the congruent numbers obtained from the squarefree parts of areas of primitive PTs not exceeding A_{top}. There is a criterion for congruent numbers by *Tunnell* (see [2], p. 3 and p. 221, or [1]) which, however, relies on the weak form of the *Birch* and *Swinnerton-Dyer* conjecture.

2. Lemmata and determination of N

We consider areas A of integer primitive PTs collected in the number triangle A249869.

$$T(n,\, m) \,=\, n\, m\, (n\,+\,m)(n\,-\,m) \;\; if\ n\,>\,m\,\geq\,1, (-1)^{n+m}\,=\,-1 \; and\; \gcd(n,m)\,=\,1,\; else\; 0\,. \tag{1}$$

In order to find from this number triangle all area entries of A024406 not exceeding a given area A_{top} we need to find a row number N which guarantees that these areas are reached for rows numbered 2, 3, ..., N. Areas larger than A_{top} are then discarded after ordering increasingly all the entries from these rows. One could try to find an optimal N, but this is not necessary.

We start with a lemma concerning the minimal entries in each row of the number triangle T.

Lemma 1: Minima in row n

a) $n\,=\,2\,k,\;k\,\geq\,1:\qquad T(2\,k,\,m)\,>\,T(2\,k,\,1),\;\;m\,=\,2\,l\,-\,1,\,l\,=\,2,\,...,\;k\,,$

b) $n\,=\,2\,k\,+\,1,\;k\,\geq\,1:\;\;T(2\,k\,+\,1,\,m)\,>\,T(2\,k\,+\,1,\,2\,k),\,m\,=\,2\,l,\,l\,=\,1,\,...,\,k\,-\,2\,.$

In the odd n case one has for $n = 3$ $(k\,=\,1)$ only the $T(3,\,2)$ entry.

Proof: *a)* Because k does not vanish this inequality means

$$(2\,l\,-\,1)\,\left((2\,k)^2\,-\,(2\,l\,-\,1)^2\right)\,-\,4\,k^2\,+\,1\,>\,0\,, \tag{2}$$

i.e.,

$$\begin{aligned}&2\,(2\,k)^2\,(l\,-\,1)\,-\,(2\,l\,-\,1)^3\,+\,1\;\geq\;2\,(2\,l)^2\,(M\,-\,1)\,-\,(2\,l\,-\,1)^3\,+\,1\,=\\&2\,(2\,l^2\,-\,3\,l\,+\,1)\,=\,2\,\left(2\,(l-2)^2+5\,(l-2)\,+\,3\right)\,>\,0,\text{ for } l\,\geq\,2\,.\end{aligned} \tag{3}$$

In the first inequality $k\,\geq\,l$ has been used.

b) This proof is a bit subtler. Dividing by $2\,k\,+\,1$ one wants to prove

$$\begin{aligned}&2\,l\,\left((2\,k\,+\,1)^2\,-\,(2\,l)^2\right)\,-\,2\,k\,(4\,k\,+\,1)\,>\,0\,,\\&k^2\,8\,(l\,-\,1)\,+\,k\,2\,(4\,l\,-\,1)\,+\,2\,l\,-\,(2\,l)^3\,>\,0\,.\end{aligned} \tag{4}$$

First we use $k^2\,\geq\,(l\,+\,1)^2$ on the *l.h.s.* (left-hand side) and then $-8\,l\,\geq\,-8\,(k\,-\,1)$.

$$\begin{aligned}&l.h.s.\,\geq\,8\,l^2\,-\,8\,l\,-\,8\,+\,2\,k\,(4\,l\,-\,1)\,+\,2\,l\,\geq\,8\,l^2\,-\,8\,(k\,-\,1)\,-\,8\,+\,2\,k\,(4\,l\,-\,1)\,+\,2\,l\\&=\,8\,l^2\,+\,2\,k\,(4\,l\,-\,5)\,+\,2\,M\,>\,0,\;\;\text{for } l\,\geq\,2\,.\end{aligned} \tag{5}$$

The case $l\,=\,1$ is treated separately, leading to $4\,k^2\,+\,4\,k\,-\,3\,>\,0,$ for $k\,\geq\,1\,.$ □

The next lemma concerns the increase of the row number by one step.

Lemma 2: Row minima increase or decrease

a) The minima increase going from row $n\,=\,2\,k$ to row $n\,=\,2\,k\,+\,1$, for $k\,\geq\,1$.

b) When going from row $n\,=\,2\,k\,+\,1$ to row $n\,=\,2\,(k\,+\,1)\,+\,1$ the minimum stays the same for $k\,=\,1$ (row $3\,\to\,4$), the minimum increases for $k\,=\,2$ (row $5\,\to\,6$), and for $k\,\geq\,3$ the minimum decreases.

Proof: a) $T(2\,k,\,1)\,<\,T(2\,k\,+\,1,\,2\,k)$. After division by $(2\,k)$ this means $(2\,k)^2\,-\,1\,=\,(2\,k\,-\,1)\,(2\,k\,+\,1)\,<\,(2\,k\,+\,1)\,(4\,k\,+\,1)$, that is, after division by $2\,k\,+\,1$, $2\,k\,+\,2\,>\,0$ which is true for the considered $k\,\geq\,1$.

b) The cases $k\,=\,1$ and 2 are clear by inspection of the table ($30\,\to\,60$ and $180\,\to\,240$). For $k\,\geq\,3$ $T(2\,k+1,\,2\,k)-T(2\,(k+1),\,1)$ becomes, after division by $2\,(2\,k+1)$, $k\,(4\,k+1)-(k+1)\,(2\,k+3)\,=\,2\,k^2\,-\,5\,k\,+\,1$, which is true for $k\,\geq\,3$. □

It is now trivial to prove the following lemma.

Lemma 3: Minima for row numbers differing by 2

a) The minimum of row number $n = 2k$ *is smaller than the one of row number* $n + 2 = 2(k+1)$*, for* $k \geq 1$.

b) The minimum of row number $n = 2k + 1$ *is smaller than the one of row number* $n + 2 = 2k + 3$*, for* $k \geq 1$.

After these preliminaries is is clear that in order to find a row number N such that all areas $\leq A_{top}$ are covered by the rows $n = 2$ to N one should look first at an even numbered row N. This is because then all entries of the table with row number $n > N$ are certainly larger than A_{top}, which would not be the case if one determines first an odd number N. It is easy to solve the cubic equation $T(x,1) = A_{top}$, *i.e.*, $x^3 - x - A_{top} = 0$. This cubic has discriminant $\triangle = \frac{1}{2^2 \cdot 3^3}(27\, A_{top}^2 - 4) > 0$, hence there is only one real solution, *i.e.*,

$$x = x(A_{top}) = \left(\frac{A_{top}}{2} + \sqrt{\triangle}\right)^{\frac{1}{3}} + \left(\frac{A_{top}}{2} - \sqrt{\triangle}\right)^{\frac{1}{3}} . \tag{6}$$

Therefore we first look at the preliminary even $N_{prel} = \lceil x(A_{top}) \rceil$ ($\lceil . \rceil$ denotes the ceiling function) if it is even or if this value is odd we look at $\lceil x(A_{top}) \rceil + 1$. *E.g.*, for $A_{top} = 209$ (which happens not to be a PT area) we have $x = 5.99063947...$, hence $N_{prel} = 6$. Because row $n = 6$ has the minimum $T(6,\, 1) = 210$, this N serves its purpose, however the minimal N would in fact be 5 because in this odd numbered row the largest area below 210, which is 180, appears as minimum $T(5,\, 4)$. Rounding errors in x can lead to an overshooting as the example of an actual area 2730 shows. Depending on precision, $x = 13.99966320$ (Maple 10 digits) but $x = 14.000000000000019780$ (Maple 20 digits). In the first case one obtains $N_{prel} = 14$ with row minimum 2730, in the second case one has 15 but the next even row number is 16, with minimum 4080. Here less is more. However, the optimal N for 2730 is 12, even though 2730 appears already in row 10 as $T(10,\, 3)$, but the rows 11 and 12 have also areas < 2730, namely 2310, 2574 and 1716, respectively. So the determination of the minimal N for a given A_{top} is a complicated matter, and we shall settle with the computed even N_{prel} as a safe N.

Proposition: Safe N

For a given positive integer A_{top} *the even row number* $N = N(A_{top})$*, chosen between*

$$\textit{either } \lceil x(A_{top}) \rceil \quad \textit{or } \; 1 + \lceil x(A_{top}) \rceil \tag{7}$$

guarantees that all primitive PT areas $A \leq A_{top}$ *are covered by the rows* $n = 2,\, 3,\, ...,\, N(A_{top})$.

As mentioned above, in this list of rows 2 to $N(A_{top})$ all areas $A > A_{top}$ have to be discarded.

In connection with the next two lists for all areas and squarefree areas up to 10^6 let us look at $A = 210$ which appears twice in the integer PT area list, and a third time in the squarefree area list. This three instances correspond to the three positions $T(5,\, 1) = 210$, $T(6,\, 2) = 210$ and $T(8,\, 7) = 840$ with the primitive PTs (21, 20, 29), (35, 12, 37) and (15, 112, 113), respectively. For the last instance the square factor of the area is 2^2 leading to the rational PT $\left(\frac{15}{2},\, 56,\, \frac{113}{2}\right)$ with area 210. For $A_{top} = 10^8$ a fourth instance of the congruent number 210 will show up for $T(128,\, 7)$ corresponding to the primitive PT (16335, 1792, 16433) with area $a = 14636160 = (2^3 \cdot 3 \cdot 11)^2 \cdot 210$, hence this rational triangle for $a = 210$ is $\left(\frac{495}{8}, \frac{224}{33}, \frac{16433}{264}\right)$.

For $A_{top} = 10^6$ with $N = 102$ we get (with multiple entries):

[6, 30, 60, 84, 180, 210, 210, 330, 504, 546, 630, 840, 924, 990, 1224, 1320, 1386, 1560, 1710, 1716, 2310, 2340, 2574, 2730, 2730, 3036, 3570, 3900, 4080, 4290, 4620, 4914, 5016, 5610, 5814, 6090, 6630, 7140, 7440, 7854, 7956, 7980, 7980, 8970, 8976, 9690, 10374, 10626, 10710, 10920, 11550, 11856, 11970, 12540, 12654, 13566, 13800, 14490, 14820, 17220, 17550, 18096, 18354, 18480, 18564, 19320, 19866, 21924, 22134, 22770, 23184, 23460, 23760, 24150, 25200, 25806, 25944, 26220, 26910, 26970, 29400, 29580, 30600, 31050, 31350, 31920, 32130, 32736, 33150, 34650, 35700, 36270, 37050, 37206, 37740, 39150, 39270, 40194, 41580, 43890, 45144, 46284, 46620, 47196, 48546, 49140, 49476, 49590, 50490, 51330, 51414, 52026, 54834, 56730, 57420, 59334, 59340, 60060, 60900, 61380, 62160, 62496, 63336, 63960, 65100, 66120, 66990, 68034, 68640, 71610, 71610, 72930, 74046, 75174, 77004, 77550, 79794, 82110, 84630, 84870, 85140, 85470, 85470, 85560, 87906, 88350, 88536, 89460, 90090, 95700, 97236, 97290, 97440, 98040, 98670, 100464, 101010, 105450, 106260, 106260, 107226, 110544, 111300, 112200, 114114, 114114, 116994, 117096, 117180, 118326, 120120, 123240, 124410, 124950, 126984, 128310, 132840, 135450, 137514, 139230, 140070, 140556, 141636, 142926, 143640, 144300, 145530, 147600, 150150, 153510, 153636, 155610, 157410, 158670, 158730, 159840, 163590, 164220, 164604, 169260, 169824, 169830, 171054, 172050, 175560, 175890, 176220, 177156, 177660, 178710, 180264, 181890, 184440, 188370, 190920, 192270, 195054, 200244, 201066, 201474, 204204, 204336, 205530, 207270, 214320, 214890, 215940, 218400, 222456, 223860, 225036, 225120, 227766, 228144, 228780, 229710, 234780, 234780, 234906, 235620, 238266, 242550, 242556, 249690, 254364, 254880, 255780, 255990, 257070, 257550, 261870, 262080, 263466, 264894, 266910, 267954, 269610, 269874, 270600, 271440, 271584, 273060, 273156, 281820, 285090, 286440, 287430, 289380, 290766, 291270, 295926, 297330, 303924, 306234, 307230, 310590, 311610, 314244, 314364, 315276, 317856, 319200, 323730, 325080, 325314, 326370, 328950, 332010, 332310, 341880, 341880, 342930, 347820, 349680, 354354, 355446, 355680, 360696, 361620, 364320, 365190, 369246, 369930, 371070, 373176, 373650, 373860, 380190, 383520, 386280, 387090, 391776, 391986, 399840, 400374, 403104, 404670, 405150, 408480, 410286, 412284, 412566, 420420, 420420, 421260, 428280, 434280, 436254, 438900, 439824, 442860, 445170, 451350, 453474, 454020, 454584, 454740, 460020, 465186, 465300, 466146, 467460, 467754, 469530, 474474, 474810, 475020, 480930, 483990, 488250, 490314, 499590, 499590, 510510, 511920, 512064, 513570, 520590, 525336, 527340, 527436, 528126, 530376, 536640, 542640, 543900, 545160, 546840, 546960, 548730, 551286, 553014, 554190, 561990, 564564, 567630, 568260, 568974, 570570, 581196, 583770, 588126, 592116, 592620, 600990, 603900, 606060, 607614, 612786, 613830, 614040, 615060, 616284, 620256, 629850, 635076, 635970, 639540, 642390, 642804, 648210, 650760, 656466, 658896, 661770, 661980, 664506, 665574, 666666, 671370, 674520, 681384, 683850, 694830, 695310, 695970, 696540, 700770, 704184, 705960, 712140, 713310, 715254, 717060, 720390, 723450, 728910, 731016, 731850, 733590, 734580, 737124, 744120, 746130, 752070, 756204, 757680, 759066, 762120, 772044, 777450, 778050, 778596, 780570, 784704, 789360, 792360, 794094, 799710, 800394, 804804, 806520, 817740, 818850, 820410, 826950, 830490, 842490, 843150, 852390, 856044, 857850, 860700, 861984, 870870, 871080, 874380, 878430, 879780, 880440, 881814, 884640, 889746, 891480, 895356, 904020, 912450, 916104, 918060, 923076, 930930, 934800, 939120, 940680, 941094, 941460, 941640, 960630, 966420, 967434, 968310, 970536, 973560, 985320, 985446, 986370, 991230, 993510, 995106, 996156, 999900].

One could go to higher A_{top}. However, the mentioned congruent number $a = 157$ for the *Zagier* triangle needs $N \approx 2.319964661 \cdot 10^{31}$, which is beyond control.

Regarding the congruent number problem the interest is in those areas which are not squarefree in order to find low congruent number for rational PTs. Therefore we close with the list of the squarefree part of primitive PTs (with multiple entries), the congruent numbers a (with multiple entries), obtained from the above list for $A_{top} = 10^6$.

[5, 6, 6, 7, 14, 15, 21, 22, 30, 34, 34, 39, 41, 65, 70, 78, 102, 110, 111, 138, 138, 141, 145, 154, 154, 161, 165, 174, 190, 205, 210, 210, 210, 221, 231, 255, 265, 286, 291, 299, 310, 330, 330, 330, 357, 390, 395, 410, 429, 434, 455, 462, 465, 470, 510, 517, 546, 546, 546, 561, 561, 602, 609, 609, 646, 651, 671, 741, 759, 798, 806, 889, 915, 957, 966, 1110, 1111, 1113, 1122, 1131, 1155, 1155, 1155, 1190, 1254, 1254, 1254, 1254, 1295, 1311, 1326, 1330, 1365, 1406, 1419, 1443, 1462, 1482, 1595, 1610, 1705, 1770, 1770, 1785, 1794, 1885, 1886, 1995, 1995, 1995, 2006, 2046, 2134, 2139, 2145, 2145, 2170, 2310, 2337, 2365, 2470, 2485, 2530, 2530, 2701, 2706, 2706, 2730, 2730, 2730, 2849, 2990, 3021, 3102, 3135, 3255, 3458, 3534, 3570, 3570, 3705, 3885, 3939, 3990, 4030, 4134, 4218, 4290, 4290, 4305, 4389, 4466, 4641, 4674, 4794, 4830, 4895, 4921, 4935, 5394, 5406, 5510, 5610, 5610, 5865, 5934, 5986, 6006, 6090, 6090, 6251, 6270, 6279, 6355, 6486, 6545, 6555, 6630, 6806, 6882, 7395, 7585, 7854, 8170, 8385, 8607, 8729, 8866, 8970, 9030, 9430, 9435, 9546, 9690, 9690, 10010, 10302, 10366, 10374, 10385, 10614, 10626, 10730, 10810, 11310, 11571, 11914, 12166, 12261, 12369, 13110, 13195, 13395, 13566, 14070, 14190, 14835, 14946, 15015, 15470, 15785, 15834, 15990, 16206, 16206, 16530, 16835, 17085, 17119, 17290, 17490, 17630, 17641, 17765, 18354, 18870, 19006, 19610, 19866, 19866, 20130, 20210, 20306, 20405, 20670, 20930, 21170, 21390, 21855, 22010, 22134, 22134, 22386, 22715, 23779, 23970, 24486, 24510, 24871, 25194, 25194, 25530, 25806, 26130, 26474, 26565, 26565, 26845, 26970, 27354, 27370, 27671, 28938, 29274, 29274, 29274, 29986, 30030, 30810, 31098, 31746, 32754, 33078, 33726, 33915, 34026, 34185, 34314, 34510, 35139, 35409, 35970, 36146, 36498, 36890, 38409, 38766, 39270, 41055, 41151, 41181, 41230, 42315, 43010, 43554, 43890, 43890, 44486, 45066, 46110, 47355, 47730, 49335, 50061, 50386, 51051, 51330, 51414, 51794, 52026, 52170, 53985, 54834, 55290, 55510, 55510, 55614, 55965, 56730, 58695, 58695, 58695, 59334, 60639, 60970, 63070, 63591, 66990, 68034, 68289, 70455, 71610, 71610, 71610, 72345, 72930, 73834, 74046, 74074, 75174, 75981, 77330, 78591, 78819, 80990, 81510, 82110, 84630, 85470, 85470, 85470, 85470, 86955, 90174, 93610, 94710, 98670, 101010, 103071, 105315, 107070, 107590, 108570, 110390, 113505, 113646, 113685, 114114, 114114, 115005, 116994, 118326, 124410, 128310, 131334, 131835, 136290, 137514, 140070, 141141, 142926, 145299, 148155, 153510, 153510, 158730, 160701, 162690, 163590, 165495, 168630, 170346, 174135, 175890, 176046, 178035, 178710, 179265, 184281, 189051, 192270, 194649, 195054, 201066, 201201, 201630, 205530, 214890, 217770, 218595, 219945, 220110, 222870, 226005, 227766, 229515, 229710, 235365, 235410, 242634, 243390, 249690, 255990, 257070, 261870, 266910, 267954, 269610, 285090, 287430, 291270, 295926, 297330, 311610, 326370, 332310, 342930, 354354, 365190, 369246, 369930, 380190, 404670, 410286, 412566, 436254, 445170, 465186, 474474, 480930, 483990, 490314, 510510, 513570, 520590, 528126, 561990, 568974, 570570, 583770, 588126, 600990, 607614, 612786, 613830, 635970, 648210, 656466, 665574, 671370, 694830, 700770, 713310, 715254, 720390, 746130, 752070, 759066, 799710, 800394, 820410, 830490, 870870, 878430, 881814, 889746, 930930, 960630, 967434, 991230, 995106].

The first missing congruent numbers from A006991 are: 13, 23, 29, 31, 37, 38, 46, 47, 53, 55, 61, 62, 69, ...

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