THEORY OF DEMLO NUMBERS

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1. In the last Physical Sciences number of the University Journal (Vol. VII, Part 3, Nov. 1938), a paper on Demlo numbers, giving their definition and some of their interesting properties, was published by one of us (D. R. Kaprekar). That paper will be referred to as D. N. I. In this paper a brief theoretical discussion of the same is undertaken.

2. NOTATION

(1) A number formed by the digits $a,b,c,\ldots\ldots k$, from left to right will be denoted by $a,b,c,\ldots\ldots k$, with intervening commas to distinguish it from their product $a.b.c.\ldots\ldots k$. (The commas are not necessary when the actual arithmetical digits are given).

(2) The number $m,m,m,\ldots\ldots$repeated n times will be denoted by m_n. m_n may form part of a bigger number, in which case, it will be separated from the rest by commas. Thus we may write 5_4 for 5555 ; or 23, 4_3, 67 for 2344467.

3. DEFINITIONS

A Demlo number D consists of three parts, a left part L, a middle part M, and a right part R, satisfying the following conditions :—

(*a*) The middle part M is of the form m_n, $(1 \leqslant m \leqslant 9)$.

(*b*) L and R have the same number of digits p. (If L has one digit less than R, a zero can be added to it on the left to satisfy this condition).

(*c*) $L+R=m_p$, m being the same digit as in (*a*), and p the number of digits in R.

Thus $D = L,m_n,R$, where $L+R=m_p$, *e.g.* 23, 7_4, 54 is a Demlo number, with $L=23$, $M=7_4=7777$, $R=54$, since $L+R=77=7_2$. Similarly 94, 6_3, 572 is a Demlo number, with $L=094$.

NOTE.—(1) The middle part may be absent altogether, as in 23, 54. Here $m=7$ but $n=0$.

(2) L and R may both be absent, in which case the number is called a linear Demlo number, *e.g.* 7_6 is a linear Demlo number. This may also be regarded as 0, 7_4, 7.

(3) If $L+R=m_{p+1}$, where p=number of digits in R, L, m_n, R is *not* a Demlo number. Thus 166,111,945 is not a Demlo number.

(4) It is unnecessary to consider forms for L, with 2 or more digits less than in R. For in such a case it is obvious that one or more left-hand digits of R will be identical with m and so can be added to m_n.

4. General Expression

A linear Demlo number $m_n=m, m, m,\ldots$ n times.

$$\therefore \quad m_n=m\times 1_n \tag{4·1}$$
$$=m\times(10^{n-1}+10^{n-2}+\ldots+10+1)$$
$$=m\times(10^n-1)/9 \tag{4·2}$$

In particular,

$$9_n=10^n-1 \tag{4·21}$$

Since a general Demlo number $D=L, m_n, R$, where $L+R=m_p$, p being the number of digits in R,

$$D=R+m_n\times 10^p+L\times 10^{n+p}$$
$$=(m_p-L)+m_n\times 10^p+L\times 10^{n+p}$$
$$=m\times(10^p-1)/9+m\times(10^n-1)\times 10^p/9+L(10^{n+p}-1)$$

by 4·2

$$=\left(L+\frac{m}{9}\right)(10^{n+p}-1) \tag{4·3}$$
$$=(9L+m)\times 1_{n+p} \tag{4·4}$$

5. Properties of Demlo Numbers

Every Demlo number has a factor of the type 1_k, where $k=n+p$=the number of digits in M, R. (5·1).

This follows immediately from 4·4.

The converse of (5·1) is true with certain limitations and may be stated as follows :—

$A\times 1_k$, where A is any number, is a Demlo number for sufficiently large values of k. (5·2).

Proof: If $A\times 1_k$ is a Demlo number L, m_n, R,

$$A=9L+m \quad \text{by (4·1)} \tag{5·3}$$

Then $m\equiv A \pmod 9,\ 1\leqslant m\leqslant 9$ (5·4)

and $L=\dfrac{A-m}{9}$. (5·5)

To find R, let the number of digits in L be q.

Then if $L<m_q$, $R=m_q-L$; otherwise $R=m_{q+1}-L$. (5·6)

If the number of digits in $m_q - L$ be less than q, R must be taken with the necessary number of zeros added on the left to equalize them.

Finally the number of digits in m_n is given by $n = k - p$ (5·7)

This shows that $k \geqslant p$; otherwise n being negative $A \times 1_k$ is not a Demlo number.

Examples.

(1) $4931 \times 1_k$.

Here $A = 4931 = 9 \times 547 + 8$.

$\therefore$ $L = 547$, $m = 8$, $R = 888 - 547 = 341$ and $p = 3$.

$\therefore$ $4931 \times 1_k$ is a Demlo number for $k \geqslant 3$.

$k = 2$ gives 5424 which is not a Demlo number.

For $k \geqslant 3$, $4931 \times 1_k = 547, 8_{k-3}, 341$, a D. N.

(2) $662 \times 1_k$.

Here $L = 073$, $m = 5$, $R = 555 - 073 = 482$ and $p = 3$.

$\therefore$ $662 \times 1_k = 73, 5_{k-3}, 482$ if $k \geqslant 3$.

(3) $7721 \times 1_k = 857, 8_{k-3}, 031$ for $k \geqslant 3$.

(4) $5175 \times 1_k = 574, 9_{k-3}, 425$ for $k \geqslant 3$.

6. Ways of obtaining Demlo Numbers

In D. N. I, a number of ways of obtaining Demlo numbers were given. It will be seen that they all reduce to the application of 5·2 in different ways, though the reservation re. the value of k is not stated there in all cases. We shall consider here only one or two cases for illustration.

(1) Processes of conical tables.

Table No. 1.	Table No. 2.
$1 \times 9 + 2 = 1$	$0 \times 9 + 8 = 8$
$12 \times 9 + 3 = 111$	$9 \times 9 + 7 = 88$
$123 \times 9 + 4 = 1111$	$98 \times 9 + 6 = 888$
... ...	... ...
$123,\ldots,k \times 9 + (k+1) = 1_{k+1}$	$987,\ldots,l \times 9 + (l-2) = 8_{11-l}$
$1 \leqslant k \leqslant 9$	$1 \leqslant l \leqslant 9$

For

$$1234,\ldots,k = 1_k + 1_{k-1} + 1_{k-2} + \ldots + 1$$
$$= (9_k + 9_{k-1} + \ldots\ldots\ldots + 9)/9$$

$$\therefore\ 123,\ldots,k \times 9 = 10^k + 10^{k-1} + \ldots + 10^1 - k \qquad \text{by } (4\cdot2)$$
$$= 1_{k+1} - k - 1$$

$$\therefore\ 1234,\ldots,k \times 9 + (k+1) = 1_{k+1} \qquad (6\cdot1)$$

Similarly $987,\ldots\ldots, l = 9_{10-l} - 123,\ldots,(9-l)$
$= 9_j - 123,\ldots\ldots\ldots(j-1)$

Where $j = 10 - l =$ number of digits in $987,\ldots, l$.

$$\therefore\ 987,\ldots\ldots, l \times 9 = 9 \times 9_j - 123,\ldots(j-1) \times 9$$
$$= 9 \times (10^j - 1) - 1_j + j \quad \text{from } (6\cdot1)$$
$$= 9,0j - 1_j + j - 9$$
$$= 8_j,9 - 1 + j - 8$$
$$\therefore\ 987,\ldots, l \times 9 + (l-2) = 8_{j+1} + j - 8 + l - 2$$
$$= 8_{11-l} \qquad (6\cdot2)$$

(2) The recurring decimal part of 1/N, where N is prime to 2 and 5, multiplied by N, is a linear Demlo number 9_n, *n* being the number of recuring digits.

Let $\frac{1}{N} = 0\cdot\dot{a},b,c,\ldots\dot{l}$, since there is no non-recurring part when N is prime to 2 and 5.

Then $\frac{10^n}{N} = a,b,c,\ldots l.\ \dot{a},b,c\ldots\dot{l}$, where n = number of the recurring digits $a, b,\ldots k$.

$$\therefore\ (10^n - 1) = N \times a, b, c\ldots l.$$
$$i.e.\ \ N \times a,b,\ldots, l = 9_n. \qquad (6\cdot3)$$

This is also obvious if we note that $1 = 0\cdot\dot{9}$.

Again if N is a prime different from 2 or 5 and the number of recurring digits is even, say 2*k*, then the recurring part itself is a Demlo number.

For

$$a,b,\ldots, l \times N = 9_{2k} = 10^{2k} - 1 \qquad \text{by } (6\cdot3)$$
$$\therefore\ a,b,\ldots, l = (10^k+1)(10^k-1)/N$$

Now N cannot be a factor of $10^k - 1$, otherwise the number of recurring digits would be k and not $2k$.

$\therefore$ N is a factor of $10^k + 1$

i.e. $\frac{10^k+1}{N}$ = an integer, A say.

$\therefore\ a,b,\ldots, l = A \times (10^k - 1) = 9A \times 1_k$, a Demlo number provided $k \geqslant p$, the number of digits in R by $(5\cdot7)$

This condition is satisfied, since $p \leqslant k$.

Corollary: If the number of digits *n* in the recurring part can be factorized *i.e.* if $n = p \times q \times \ldots r$ say, then $a,b,\ldots, l$ is divisible by 9_p ; 9_q ;...9_r.

***e.g.*, 032258064516129 the recurring part of $\frac{1}{31}$ contains 15 digits and so is divisible by 999 and 99999.**

7. Wonderful Demlo Numbers

A number of the type

$$123, \ldots, k, \ (k-1), \ (k-2), \ldots, \ 321, \ 1 < k \leqslant 9,$$

where the digits ascend and descend by differences of **1**, is called a Wonderful Demlo number.

A wonderful Demlo number $W = 123, \ldots, k, \ldots 321$ is a perfect square $= (1_k)^2$.

A2477

Here $L = 123, \ldots, k-1, \ m = k, \ p = k-1, \ n = 1.$

$$\therefore \quad W = (9L + m) \times 1_{n+p} \qquad \text{by } (4 \cdot 4)$$

$$= \{9 \times 123, \ldots, (k-1) + k\} \times 1_k \quad \text{since } n + p = k$$

$$= 1_k \times 1_k \qquad \text{by } (6 \cdot 1)$$

$$\therefore \quad W = (1_k)^2 \qquad (7 \cdot 1)$$

A most striking property of these numbers was stated in D. N. I and is proved in a paper by one of the authors (D. R. K.) in the Mathematics Student, Vol. 6, June 1938. It is proved here again by a different method.

Partition and insertion property

Dissectible numbers. This is a set of numbers, of the form $a, b, c,$ such that when any one of them is multiplied by a Wonderful Demlo number, the product takes the form $a, x_s, b, y_s, c.$ (a or b or both may be zero)

e.g. $162 \times 121 = \mathbf{1}, 9, \mathbf{6}, 0, \mathbf{2}$; $162 \times 12321 = \mathbf{1}, 99, \mathbf{6}, 00, \mathbf{2}.$

We shall now investigate the conditions to be satisfied by dissectible numbers.

For simplicity let us consider $1234321 \times a, b, c,$ a, b, c being a dissectible number.

By the usual form of the operation of multipliction

$$\begin{array}{r} 1234321 \\ \times \ a,b,c \end{array}$$

$$\begin{array}{r} c, \ (2c), \ (3c), \ (4c), \ (3c), \ (2c), \ c \\ b, \ (2b), \ (3b), \ (4b), \ (3b), \ (2b), \ b \quad\quad \\ a, \ (2a), \ (3a), \ (4a), \ (3a), \ (2a), \ a \quad\quad\quad\quad \end{array}$$

$$a, (2a+b), (3a+2b+c), (4a+3b+2c), (3a+4b+3c), (2a+3b+4c), (a+2b+3c), (b+2c), c.$$

Here the terms in the brackets may have more than one digit, but they have their place value.

If a, b, c is a dissectible number,

(i) $2a+b \not> 9$ for otherwise the left-hand-most digit of the product will differ from a. It will be seen later that $2a+b \leqslant 8$.

(ii) $b+2c>9$.

For if $b+2c \leqslant 9$, since 3rd digit from the right must be $b+2c$, $a+2b+3c=b+2c+10$

i.e. $a+b+c=10$, requiring 1 to be carried over.

Then 4th digit from right$=2a+3b+4c+1-20$

$=b+2c+1 \neq b+2c$

$\therefore \quad b+2c>9.$ (7·2)

Let $b+2c=10+y, \quad 0 \leqslant y \leqslant 9$

Then the 2nd digit from the right$=y$

" " 3rd ....

$=a+2b+3c+1-$greatest multiple of ten

$=y+10+a+b+c+1-$g. m. t.

$=y$ if $a+b+c=9$ or 19

carrying over 2 if $a+b+c=9$

and 3 if $a+b+c=19$

If $a+b+c=19$

Then 4th digit$=2a+3b+4c+3-$g. m. t.

$=b+2c+38+3-$g. m. t.

$=y+1+40-$g. m. t.

$\neq y$.

$\therefore \quad a+b+c=9$ (7·3)

Then 4th digit$=2a+3b+4c+2-$greatest multiple of ten.

$=b+2c+20-$g. m. t.

$=y+30-$g. m. t.

$=y$ carrying over 3.

Then 5th digit$=3a+4b+3c+3-$g. m. t.

$=b+30-$g. m. t.

$=b$ carrying over 3.

Then 6th digit$=4a+3b+2c+3-$g. m. t.

$=2a+b+1+20-$g. m. t.

$=2a+b+1$ carrying over 2 if $2a+b \leqslant 8$.

Then 7th digit$=3a+2b+c+2-$g. m. t.

$=2a+b+1+10-$g. m. t.

$=2a+b+1$ carrying over 1.

Then 8th digit$=2a+b+1$

and 9th digit$=a$

$\therefore \quad$ Product$=a, (2a+b+1)_3, b, (b+2c-10)_3, c$

If $2a+b=9$, then 6th digit$=0$ carrying over 3.

and 7th digit$=3a+2b+c+3-$g. m. t.

$=21-$g. m. t.

$=1\neq$6th digit.

$$\therefore \quad 2a+b\leqslant 8. \tag{7·4}$$

It can be seen that the same conditions must be satisfied for the product by any wonderful Demlo number.

Thus a dissectible number a,b,c satisfies the following conditions:—

(i) $a+b+c=9$;

(ii) $b+2c>9$ *i.e.* $a<c$ from (i);

(iii) $2a+b\leqslant 8$ or $c\geqslant a+1$ and $a\leqslant 4$.

We thus obtain the following 25 dissectible numbers only.

A249605 009; 018; 027; 036; 045; 054; 063; 072; 081;
108; 117; 126; 135; 144; 153; 162;
207; 216; 225; 234; 243;
306; 315; 324;
405;

A249605

When a,b,c is multiplied by any wonderful Demlo numbers, the product,

$$123,\ldots\ldots k,\ k-1,\ldots\ldots,321 \times a,b,c$$
$$=a,\ (2a+b+1)_{k-1},\ b,\ (b+2c-10)_{k-1},\ c.$$

This discussion is sufficient to show the great variety of interesting properties possessed by Demlo numbers. Other properties will be considered theoretically later on.

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