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Notations

$n = 2^m \times q$, $m \geq 0$ and q odd.
S(n, k) denotes the k-th entry in row n of triangle A237048,
T(n, k) denotes the k-th entry in row n of triangle A249223,
each row having $r = \text{row}(n) = \left\lfloor \frac{\sqrt{(8n+1)} - 1}{2} \right\rfloor$ entries.
S(n, k) = 1 if k is odd and k|n or if k is even and $k \mid \left(n - \frac{k}{2}\right)$, 0 otherwise.

Theorem

$T(n, k) = |\{ d: d|n \;\&\; \frac{k}{2} < d \leq k \}|$, $1 \leq k \leq \text{row}(n)$.

Proof

$k \mid \left(n - \frac{k}{2}\right) \Leftrightarrow \left(2^m \times q = \frac{k}{2} \times (2 \times s + 1), \text{ for suitable } s \right) \Leftrightarrow k = 2^{m+1} \times d$ with $d|q$.

T(n, k)
$= \sum_{i=1}^{r} (-1)^{i+1} S(n, k)$
$= \sum_{d \mid q \,\&\, d \leq k} 1 - \sum_{d \mid q \,\&\, (2^{m+1} \times d) \leq k} 1$ (*)
$= |\{d : d \mid q, d \leq k\}| - |\{2^{m+1} \times d : d \mid q,\ 2^{m+1} \times d \leq k\}|$
$= |\{d : d \mid q, d \leq k\} \cup \{2^j \times d : 1 \leq j \leq m,\ d \mid q,\ 2^j \times d \leq k\}| -$
$|\{2^{m+1} \times d : d \mid q,\ 2^{m+1} \times d \leq k\} \cup \{2^j \times d : 1 \leq j \leq m,\ d \mid q,\ 2^j \times d \leq k\}|$
$= |\{2^j \times d : 0 \leq j \leq m,\ d \mid q,\ 2^j \times d \leq k\}| - |\{2^j \times d : 1 \leq j \leq m+1,\ d \mid q,\ 2^j \times d \leq k\}|$
$= |\{2^j \times d : 0 \leq j \leq m,\ d \mid q,\ 2^j \times d \leq k\}| - |\{2^j \times d : 0 \leq j \leq m,\ d \mid q,\ 2^j \times d \leq \frac{k}{2}\}|$
$= |\{2^j \times d : 0 \leq j \leq m,\ d \mid q,\ \frac{k}{2} < 2^j \times d \leq k\}|$
$= |\{d : d \mid n,\ \frac{k}{2} < d \leq k\}|$

Corollary

The width of the symmetric representation of sigma of n at the diagonal, T(n, row(n)), equals the number of divisors of n in the half-open interval $\left(\frac{\text{row}(n)}{2},\ \text{row}(n)\right]$.

Proof

Line (*) says that the number of odd divisors of n less than or equal to row(n) minus the number of odd divisors of n greater than row(n) equals T(n, row(n)), the width of the symmetric representation of sigma of n at the diagonal.