Electrical Capacitance

Steven Finch

August 25, 2014

We mentioned **logarithmic capacity** or **transfinite diameter** in [1]. Given a compact set A in $\mathbb{R}^2$, the measure

$$\gamma_0(A) = \lim_{n\to\infty} \max_{\xi_1,\ldots,\xi_n\in A} \left(\prod_{j<k} \left|\xi_j - \xi_k\right| \right)^{\frac{2}{n(n-1)}}$$

is invariant under rigid motions and continuous, but fails to be additive since $\gamma_0(A) = \gamma_0(\partial A)$ [2, 3, 4]. The unit interval has logarithmic capacity $1/4$; the unit disk, square and equilateral triangle have logarithmic capacities

$$1, \qquad \frac{1}{4\pi^{3/2}}\Gamma\left(\frac{1}{4}\right)^2 = 0.5901702995..., \qquad \frac{\sqrt{3}}{8\pi^2}\Gamma\left(\frac{1}{3}\right)^3 = 0.4217539346...$$

respectively. Discussion of the geometric mean (of all pairs of points) often seems to be restricted to planar sets; we now turn to the harmonic mean and subsequently to the arithmetic mean.

Given a compact set A in $\mathbb{R}^3$, define [5, 6]

$$\gamma_{-1}(A) = \lim_{n\to\infty} \max_{\xi_1,\ldots,\xi_n\in A} \left(\frac{2}{n(n-1)} \sum_{j<k} \frac{1}{\left|\xi_j - \xi_k\right|} \right)^{-1}$$

to be the **Newtonian capacity** or **electrical capacitance** or **generalized transfinite diameter** of order -1. This is also the reciprocal of what is known as the **optimal Riesz 1-energy** [7]. The unit interval and unit circle both have electrical capacitance 0; one way to see the latter is to notice the inequality [8]

$$\sum_{j<k} \frac{1}{\left|\xi_j - \xi_k\right|} \geq \frac{n}{4} \sum_{\ell=1}^{n-1} \csc\left(\frac{\ell\pi}{n}\right)$$

(for which equality holds when $\xi_1, \ldots, \xi_n$ are n^{th} roots of unity). The unit disk has capacitance $2/\pi$ [9] If A is the closure of a bounded, open, connected set in $\mathbb{R}^3$, then

$\gamma_{-1}(A) = \gamma_{-1}(\partial A)$ [10]. The unit ball (and hence the unit sphere) has capacitance 1. Another way to see this is to invoke a formula for s-energy of the d-sphere [11] with $s = 1$, $d = 2$.

Interesting constants arise here. For example, let A be the solid formed by revolving a disk of radius 1 about a tangent line (a "torus without hole"). It follows that [12]

$$\gamma_{-1}(A) = \frac{4}{\pi}\int_{0}^{\infty} \frac{1}{I_0(t)^2}dt = 4\,(0.4353450662...)$$

where $I_0(t)$ is the zeroth modified Bessel function. More generally, consider the surface formed by revolving an arc of a circle about its chord (a "spindle"). A definite integral involving Legendre functions of complex degree, parametrized by the included angle, is found [13]. As another example, consider the (disconnected) set consisting of two congruent parallel line segments. Its capacitance is obtained via a transcendental equation that involves elliptic integrals [14, 15, 16]. See [10, 17, 18, 19, 20] for more examples.

Seemingly simple sets present formidably difficult challenges [21]. The unit cube C has attracted enormous attention [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41] and the best numerical estimate is [2, 9, 42]

$$\gamma_{-1}(C) = 0.6606781540... = \frac{1}{2}(1.3213563081...).$$

A conjectured exact expression for $\gamma_{-1}(C)$ in [43, 44] is evidently incorrect. For the unit square S and the unit equilateral triangle T, we have less precision:

$$\gamma_{-1}(S) = 0.366789... = \frac{1}{2}(0.733579...) = \frac{2}{\pi}(0.576151...),$$

$$\gamma_{-1}(T) = 0.2508... = \frac{2}{\pi}(0.3940...).$$

It would be good someday to see improvements of these estimates, as well as $0.3565... = (1.7465...)/\sqrt{24}$ for the unit regular tetrahedron. We wonder if formulation in [45, 46] might assist in accomplishing this.

The preceding results are dimensionless, of course. Certain authors chose to express their estimates in the following manner:

$$\gamma_{-1}(C) \approx \frac{1}{4\pi\varepsilon_0}(73.51036),$$

$$\gamma_{-1}(S) \approx \frac{1}{4\pi\varepsilon_0}(40.811) \approx \frac{1}{\sqrt{2}}\frac{1}{4\pi\varepsilon_0}(57.715),$$

$$\gamma_{-1}(T) \approx \frac{1}{4\pi\varepsilon_0}(27.91) \approx \frac{1}{\sqrt{3}}\frac{1}{4\pi\varepsilon_0}(48.33)$$

where $4\pi\varepsilon_0 \approx 111.265006$ picofarads/meter and ε_0 is the permittivity constant of free space. Such decisions are a little unfortunate for us, since the value of ε_0 is based on physical experimentation and thus the normalization has changed somewhat with the passage of time.

Moving back to geometry, define the **generalized transfinite diameter** of order 1 or **optimal Riesz** (-1)**-energy**

$$\gamma_1(A) = \lim_{n\to\infty} \max_{\xi_1,\ldots,\xi_n\in A} \left(\frac{2}{n(n-1)} \sum_{j<k} \left|\xi_j - \xi_k\right| \right)$$

where A is a compact set in $\mathbb{R}^3$ [5, 7]. For lack of a convenient phrase ("sums of distances" is vague), we call $\gamma_1(A)$ the **Euclidean capacity** of A. The unit interval has Euclidean capacity 1/2. The unit disk (and hence the unit circle) has Euclidean capacity $4/\pi$; notice the inequality [8]

$$\sum_{j<k} \left|\xi_j - \xi_k\right| \leq n \cot\left(\frac{\pi}{2n}\right)$$

(for which equality holds when $\xi_1, \ldots, \xi_n$ are n^{th} roots of unity). The unit ball (and hence the unit sphere) has Euclidean capacity 4/3; set $s = -1$, $d = 2$ in the formula for s-energy of the d-sphere [11]. We wrote 2/3 in [47] since sums were divided by n^2 rather than $2/(n(n-1))$. Higher order asymptotics for the latter are conjectured in [48].

It is remarkable that no numerical results for Euclidean capacity (akin to those for Newtonian capacity) of the unit cube, square, equilateral triangle or regular tetrahedron appear yet to exist. A starting point for a literature search might be [49, 50, 51, 52, 53, 54].

References

[1] S. R. Finch, Integer Chebyshev constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 268–272.

[2] T. Ransford, Computation of logarithmic capacity, *Comput. Methods Funct. Theory* 10 (2010) 555–578; MR2791324 (2012c:65047).

[3] W. K. Hayman, *Subharmonic Functions.* v. II, Academic Press, 1989, pp. 429–436; MR1049148 (91f:31001).

[4] P. Pyrih, Logarithmic capacity is not subadditive – a fine topology approach, *Comment. Math. Univ. Carolin.* 33 (1992) 67–72; MR1173748 (93g:31002).

[5] G. Pólya and G. Szegö, Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen, *J. Reine Angew. Math.* 165 (1931) 4–49.

[6] L. Carleson, *Selected Problems on Exceptional Sets*, Van Nostrand Co., 1967, pp. 37–39; MR0225986 (37 #1576).

[7] J. S. Brauchart, Optimal discrete Riesz energy and discrepancy, *Unif. Distrib. Theory* 6 (2011) 207–220; MR2904048.

[8] L. Fejes Tóth, On the sum of distances determined by a pointset, *Acta Math. Acad. Sci. Hungar.* 7 (1956) 397–401; MR0107212 (21 #5937).

[9] Q. Rajon, T. Ransford and J. Rostand, Computation of capacity via quadratic programming, *J. Math. Pures Appl.* 94 (2010) 398–413; MR2719894 (2011h:31011).

[10] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, 1972, pp. 160–177; MR0350027 (50 #2520).

[11] J. S. Brauchart and P. J. Grabner, Distributing many points on spheres: minimal energy and designs, *J. Complexity* 31 (2015) 293–326; arXiv:1407.8282; MR3325677.

[12] C. J. Bouwkamp, A simple method of calculating electrostatic capacity, *Physica* 24 (1958) 538–542; MR0074301 (17,563c).

[13] V. P. Pyati, Capacitance of a spindle, *Proc. IEEE* 61 (1973) 505–506.

[14] I. B. Simonenko and A. A. Čekulaeva, The capacity of a condenser that consists of infinite bands (in Russian), *Izvestija Vysših Učebnyh Zavedeniĭ. Èlektromehanika* 4 (1972) 362–365; MR0304684 (46 #3816).

[15] D. Karp, Capacity of a condenser whose plates are circular arcs, *Complex Var. Theory Appl.* 50 (2005) 103–122; arXiv:math/0604033; MR2122748 (2005j:31001).

[16] V. N. Dubinin and D. Karp, Capacities of certain plane condensers and sets under simple geometric transformations, *Complex Var. Elliptic Equ.* 53 (2008) 607–622; MR2421820 (2010e:31001).

[17] E. J. Routh, *A Treatise on Analytical Statics*, v. II, Cambridge Univ. Press, 1922, pp. 219–224.

[18] W. R. Smythe, *Static and Dynamic Electricity*, McGraw-Hill, 1950, pp. 111–123.

[19] C. Snow, *Formulas for Computing Capacitance and Inductance*, National Bureau of Standards circular #544, US Government Printing Office, 1954.

[20] Y. Y. Jossel, E. S. Kochanov and M. G. Strunskij, *The Calculation of Electrical Capacitance*, Energija, 1969 (in Russian); English transl. at http://www.dtic.mil/cgi-bin/GetTRDoc?AD=AD0727198.

[21] G. Pólya, Estimating electrostatic capacity, *Amer. Math. Monthly* 54 (1947) 201–206; MR0020175 (8,514a).

[22] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton Univ. Press, 1951, pp. 76–78; MR0043486 (13,270d).

[23] D. K. Reitan and T. J. Higgins, Calculation of the electrical capacitance of a cube, *J. Appl. Phys.* 22 (1951) 223–226; MR0040505 (12,704g).

[24] D. K. Reitan and T. J. Higgins, Accurate determination of the capacitance of a thin rectangular plate, *AIEE Trans. Part I. Communication and Electronics* 75 (1957) 761–766.

[25] B. Noble, The numerical solution of the singular integral equation for the charge distribution on a flat rectangular lamina, *Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-Differential Equations*, Rome, 1960, Birkhäuser, 1960, pp. 530–543; MR0129149 (23 #B2186).

[26] B. Noble, Some applications of the numerical solution of integral equations to boundary value problems, *Conference on Applications of Numerical Analysis*, Dundee, 1971, Lect. Notes in Math. 228, Springer-Verlag, 1971, pp. 137–154; MR0353711 (50 #6194).

[27] D. Greenspan and E. Silverman, Approximate solution of the exterior Dirichlet problem and the calculation of electrostatic capacity, *Riv. Mat. Univ. Parma* 7 (1966) 305–314; MR0233522 (38 #1843).

[28] D. Greenspan. Resolution of classical capacity problems by means of a digital computer, *Canad. J. Phys.* 44 (1966) 2605–2614.

[29] E. E. Okon and R. F. Harrington, A method of computing the capacitance of flat discs of arbitrary shape, *Electromagnetics* 1 (1981) 229–241.

[30] E. F. Kuester, Explicit approximation for the static capacitance of a microstrip patch of arbitrary shape, *J. Electromagnetic Waves Appl.* 2 (1988) 103–135.

[31] C. S. Brown, Capacity of the regular polyhedra, *Comput. Math. Appl.* 20 (1990) 43–56; MR1051753 (91f:65198).

[32] H. J. Wintle, The capacitance of the regular tetrahdron and equilateral triangle, *J. Electrostatics* 26 (1991) 115–200.

[33] H. J. Wintle, Maxwell and the boundary element method: A historical puzzle, *IEEE Electrical Insulation Magazine* 14 (1998) 23–25.

[34] H. J. Wintle, The capacitance of the cube and square plate by random walk methods, *J. Electrostatics* 62 (2004) 51–62.

[35] E. Goto, Y. Shi and N. Yoshida, Extrapolated surface charge method for capacity calculation of polygons and polyhedra, *J. Comput. Phys.* 100 (1992) 105–115.

[36] F. H. Read, Improved extrapolation technique in the boundary element method to find the capacitances of the unit square and cube, *J. Comput. Phys.* 133 (1997) 1–5.

[37] F. H. Read, Capacitances and singularities of the unit triangle, square, tetrahedron and cube, *COMPEL* 23 (2004) 572–578.

[38] E.-W. Bai and K. E. Lonngren, On the capacitance of a cube, *Computers and Electrical Engineering* 28 (2002) 317–321.

[39] M. Mascagni and N. A. Simonov, The random walk on the boundary method for calculating capacitance, *J. Comput. Phys.* 195 (2004) 465–473.

[40] S. Ghosh and A. Chakrabarty, Estimation of capacitance of different conducting bodies by the method of rectangular subareas, *J. Electrostatics* 66 (2008) 142–146.

[41] C.-O. Hwang, M. Mascagni, and T. Won, Monte Carlo methods for computing the capacitance of the unit cube, *Math. Comput. Simulation* 80 (2010) 1089–1095; MR2610070.

[42] J. Helsing and K.-M. Perfekt, On the polarizability and capacitance of the cube, *Appl. Comput. Harmon. Anal.* 34 (2013) 445–468; arXiv:1203.5997; MR3027912.

[43] J. B. Hubbard and J. F. Douglas, Hydrodynamic friction of arbitrarily shaped Brownian particles, *Phys. Rev. E* (1993) R2983–R2986.

[44] J. F. Douglas, H.-X. Zhou and J. B. Hubbard, Hydrodynamic friction and the capacitance of arbitrarily shaped objects, *Phys. Rev. E* (1994) 5319–5331.

[45] S. S. Vinogradov, P. D. Smith and E. D. Vinogradova, *Canonical Problems in Scattering and Potential Theory.* v. I, *Canonical Structures in Potential Theory*, Chapman & Hall/CRC, 2001, pp. 291–309; MR1851593 (2002f:78009).

[46] A. G. Ramm, A variational principle and its application to estimating the electrical capacitance of a perfect conductor, *Amer. Math. Monthly* 120 (2013) 747–751; MR3096484.

[47] S. R. Finch, Tammes' constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 508–511.

[48] J. S. Brauchart, D. P. Hardin and E. B. Saff, The next-order term for optimal Riesz and logarithmic energy asymptotics on the sphere, *Recent Advances in Orthogonal Polynomials, Special Functions, and Their Applications*, ed. J. Arvesú and G. López Lagomasino, Amer. Math. Soc., 2012, pp. 31–61; arXiv:1202.4037; MR2964138.

[49] E. Hille, Some geometric extremal problems, *J. Austral. Math. Soc.* 6 (1966) 122–128; MR0198349 (33 #6507).

[50] R. Alexander and K. B. Stolarsky, Extremal problems of distance geometry related to energy integrals, *Trans. Amer. Math. Soc.* 193 (1974) 1–31; MR0350629 (50 #3121).

[51] B. Meyer, Inequalities for the generalized transfinite diameter, *Rocky Mountain J. Math.* 11 (1981) 267–270; MR0619675 (82g:31011).

[52] E. Alarcon and K. B. Stolarsky, Local diameters of compact planar sets, *J. Geom.* 55 (1996) 5–22; MR1377433 (96m:52003).

[53] X. Hou and J. Shao, Spherical distribution of 5 points with maximal distance sum, *Discrete Comput. Geom.* 46 (2011) 156–174; arXiv:0906.0937; MR2794362 (2012e:52005).

[54] R. Nerattini, J. S. Brauchart and M. K.-H. Kiessling, "Magic" numbers in Smale's 7th problem, arXiv:1307.2834.