Notations

We use the definition of $T(n, k) = \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil$ for $1 \le n$ and $1 \le k \le \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right)\right\rfloor = \text{row}(n)$, and $S(n, k) = T(n, k) - T(n, k + 1)$ from A237591 and A237593, respectively. Observe that $T(n, 1) = n$.

Lemma

Let $n, i, p \in \mathbb{N}$ where $i \ge 1$ and $p \ge 3$. Equivalent are:

(1) $n = 2^{i-1} \times p^2$, where $2^i < p$ and p is a prime.

(2) (i) $2^i < p \le \text{row}(n) < 2^{\left\lceil \frac{i}{2} \right\rceil} \times p \le 2^i \times p$,

(ii) $T(n, 2^i) = T(n - 1, 2^i) + 1$,

(iii) $T(n, p) = T(n - 1, p) + 1$,

(iv) for all $k \ne 2^i, p$ with $1 < k \le \text{row}(n)$, $T(n, k) = T(n - 1, k)$,

(v) $T(n, 1) - T(n, 2^i) = \frac{1}{2} \times (2^i - 1) \times (p^2 + 1)$.

Theorem

The symmetric representation of $\sigma(n)$ consists of three regions of width 1 where the two extremal regions each have $2^k - 1$ legs and the central region starts with the p-th leg of the associated Dyck path for $\sigma(n)$ precisely when $n = 2^{k-1} \times p^2$ where $n, k, p \in \mathbb{N}$, $k \ge 1$, $p \ge 3$ is a prime, and $2^k < p \le \text{row}(n)$. Furthermore, the areas of the two outer regions are $\frac{1}{2} \times (2^k - 1) \times (p^2 + 1)$ each so that the area of the central region is $(2^k - 1) \times p$.

Proof of Lemma "(1) ⇒ (2)"

Properties (2.i), (2.ii), (2.iii) and (2.v) are easily established by direct computations.

(2.iv) For any $1 < k \le \text{row}(n)$, let $n = q \times k + d$ with $q, k, d \in \mathbb{N}$ and $0 \le d < k$.
If k is odd and $k \ne p$, then k does not divide n so that $0 < d < k$, i.e., $T(n, k) = T(n - 1, k)$.
If k is even and $k \ne 2^i$, then the three cases $0 < d < \frac{k}{2}$, $\frac{k}{2} < d < k$ and $d = \frac{k}{2}$ need to be considered. The first two follow from direct computations; when $d = \frac{k}{2}$ then $2 \times n = 2^i \times p^2 = (2 \times q + 1) \times k$ so that $k = 2^i, 2^i \times p$ or $2^i \times p^2$. The first possibility contradicts the assumption on k, and the other two cannot occur since by condition (2.i) $k \le \text{row}(n) < 2^{\left\lceil \frac{i}{2} \right\rceil} \times p \le 2^i \times p < 2^i \times p^2$.
This establishes $T(n, k) = T(n - 1, k)$ for all $k \ne 2^i, p$ with $1 < k \le \text{row}(n)$.

Proof of Lemma "(2) ⇒ (1)"

First, observe that assumption (2.i) insures that $T(n, 2^i)$ and $T(n, p)$ are well defined.

Suppose that $n = q \times 2^{i-1} + d$ with $q, d \in \mathbb{N}$ and $0 \le d < 2^{i-1}$. From assumption (2.ii) we get:

$$T(n,\ 2^i) = \left\lceil \frac{q \times 2^{i-1} + d + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d+1}{2^i} \right\rceil - 2^{i-1}$$

$$T(n-1,\ 2^i) = \left\lceil \frac{q \times 2^{i-1} + d - 1 + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d}{2^i} \right\rceil - 2^{i-1}$$

so that $\left\lceil \frac{q-1}{2} + \frac{d+1}{2^i} \right\rceil = \left\lceil \frac{q-1}{2} + \frac{d}{2^i} \right\rceil + 1$.

If q is odd, then $\left\lceil \frac{d+1}{2^i} \right\rceil = \left\lceil \frac{d}{2^i} \right\rceil + 1$, so that $d = 0$.

If q is even, then $\frac{q}{2} + \left\lceil \frac{-1}{2} + \frac{d+1}{2^i} \right\rceil = \frac{q}{2} + \left\lceil \frac{-1}{2} + \frac{d}{2^i} \right\rceil + 1$ requires $d = 2^{i-1}$, a contradiction.

Therefore, 2^{i-1} is a divisor of n.

If 2^i divides n, say $n = z \times 2^i$ for some $z \in \mathbb{N}$, then

$T(n,\ 2^i) = \left\lceil \frac{z \times 2^i + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = z - 2^{i-1} + \left\lceil \frac{-1}{2} + \frac{1}{2^i} \right\rceil = z - 2^{i-1}$

$T(n-1,\ 2^i) = \left\lceil \frac{z \times 2^i - 1 + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil = z - 2^{i-1} + \left\lceil \frac{-1}{2} \right\rceil = z - 2^{i-1}$

which contradicts (2.ii) so that 2^{i-1} is the largest power of two dividing n.

Similarly, let $n = a \times p + b$ with $a, b \in \mathbb{N}$ and $0 \le b < p$ for an odd number p. Then the expressions

$T(n, p) = \left\lceil \frac{a \times p + b + 1}{p} - \frac{p+1}{2} \right\rceil = a - \frac{p+1}{2} + \left\lceil \frac{b+1}{p} \right\rceil$

$T(n - 1, p) = \left\lceil \frac{a \times p + b - 1 + 1}{p} - \frac{p+1}{2} \right\rceil = a - \frac{p+1}{2} + \left\lceil \frac{b}{p} \right\rceil$

imply with (2.iii) that $\left\lceil \frac{b+1}{p} \right\rceil = \left\lceil \frac{b}{p} \right\rceil + 1$ so that $b = 0$.

Therefore, p is a divisor of n. If n has a prime divisor $k \le row(n)$ with $k \neq p$ then assumption (2.iv), $T(n, k) = T(n - 1, k)$, implies $\left\lceil \frac{n+1}{k} \right\rceil = \left\lceil \frac{n}{k} \right\rceil$ which is a contradiction. Therefore, p is the only odd divisor less than row(n) and is a prime.

Finally suppose that $n = s \times p \times 2^{i-1}$ with $1 < s \in \mathbb{N}$. Then we get:

$T(n, 1) - T(n,\ 2^i) = s \times p \times 2^{i-1} - \left\lceil \frac{s \times p \times 2^{i-1} + 1}{2^i} - \frac{2^i + 1}{2} \right\rceil$

$= s \times p \times 2^{i-1} - \frac{s \times p - 1}{2} + 2^{i-1} - \left\lceil \frac{1}{2^i} \right\rceil = \frac{1}{2} \times \left(2^i \times s \times p - s \times p + 1 + 2^i - 2\right)$

$= \frac{1}{2} \times \left(\left(2^i - 1\right) \times s \times p + \left(2^i - 1\right)\right) = \frac{1}{2} \times \left(2^i - 1\right) \times (s \times p + 1)$.

Now condition (2.v) leads to the following equation:

$\frac{1}{2} \times \left(2^i - 1\right) \times (s \times p + 1) = \frac{1}{2} \times \left(2^i - 1\right) \times \left(p^2 + 1\right)$.

In other words, s = p, and $n = 2^{i-1} \times p^2$.

Proof of Theorem

The lengths of the segments in the symmetric Dyck paths that bound the first half of the symmetric representation of $\sigma(n)$ are given by:

$S(n, k) = T(n, k) - T(n, k + 1)$ for $1 \le n$ and $1 \le k \le row(n)$.

The three conditions (2.ii), (2.iii) & (2.iv) together with $T(n, 1) = n$ and $2^i < p$ from condition (2.i) imply that the first region has width 1 and continues through $S\left(n,\ 2^i - 1\right)$ and that the second region starts with leg $S(n, p)$, continues through the diagonal and also has width 1.

The difference $T(n, 1) - T(n, 2^i)$ equals the area of the first region so that the area of the central region is $\left(2^k - 1\right) \times p$.