In how many ways can we place two dominoes on the n x n chessboard?

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There are three cases to consider: both dominoes horizontal, both vertical, and mixed. With same orientation the treatment is identical in both cases, so

$$A_n = \mathrm{card}(\mathbf{S}_{\text{all}}) = \\ 2 \cdot \mathrm{card}(\mathbf{S}_{\text{same or.}}) + \mathrm{card}(\mathbf{S}_{\text{mixed or.}})$$

Starting with the same orientation case, the dominoes can be on the same line or different lines. The contribution from different lines is $n(n-1)^3/2$ because there are $(n-1)^2$ ways to place two dominoes on two different lines of length n, and there are $n(n-1)/2$ combinations of two lines out of n. Secondly, if the dominoes are on the same line, there are $(n-2)(n-3)/2$ possibilities to place them, and n lines, so we have

$$\mathrm{card}(\mathbf{S}_{\text{same or.}}) = \frac{1}{2}(n^4 - 2n^3 - 2n^2 + 5n).$$

Regarding differently orientated dominoes the area unaffected by the first domino (fixed as vertical) consists of $n-2$ complete rows and two incomplete rows, the size of the latter being dependent on the position of the first domino. The first contribution amounts to $n(n-1)^2(n-2)$. The second is twice the number of ways a monomer and a dimer can be placed on a strip of length n, times the number of row placements of the first domino, so this is $2(n-1)^2(n-2)$. Therefore,

$$\mathrm{card}(\mathbf{S}_{\text{mixed or.}}) = n^4 - 2n^3 - 3n^2 + 8n - 4.$$

So finally,

$$A_n = 2n^4 - 4n^3 - 5n^2 + 13n - 4.$$