Characterization and Formula for A241010:
Numbers n with the property that the number of parts in the symmetric representation of σ(n) is odd , and that all parts have width 1.

Hartmut F. W. Höft
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All references to notations, lemmas and theorems can be found in the link of A241561 mentioned above. The proofs of Lemmas A & B and the Theorem closely follow those of Lemmas 6 & 7 and Theorem 6 stated in the link cited above.

LEMMA A:

Let n = $2^m \times$ q = $2^m \times \prod_{i=1}^{k} p_i^{e_i}$ with m ≥ 0, k ≥ 0, 2 < p_1 < … < p_k primes, and $e_i \in \mathbb{N}$, $e_i \geq 1$, for all 1 ≤ i ≤ k, be the prime factorization of n. Suppose that for all 1 ≤ i ≤ k, e_i is even and that for any two odd divisors f < g of n, $2^{m+1} \times$ f < g. Then $c_n = \sigma_0(q)$ is odd and $w_n = 1$.

PROOF:

Since every e_i, 1 ≤ i ≤ k, is even we get $\sigma_0(q) = \sigma_0(\prod_{i=1}^{k} p_i^{e_i}) = \prod_{i=1}^{k}(e_i + 1)$ is odd. Suppose that the odd divisors of n are $1 = d_1 < \ldots < d_x < d_{x+1} < \ldots < d_{2x+1}$ = q where $2 \times x+1 = \sigma_0(q)$. Then $d_y \times d_{2x+2-y}$ = q, for all 1 ≤ y ≤ x. By Lemma 1(e) the odd divisors d_{2x+1-y}, 1 ≤ y ≤ x, are represented by 1's in positions $2^{m+1} \times d_y$ in the n-th row of irregular triangle A237048. Therefore, the condition $2^{m+1} \times$ f < g for any two odd divisors implies that 1's in odd and even positions alternate in that row and $w_n = 1$.

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LEMMA B:

Let n = $2^m \times$ q = $2^m \times \prod_{i=1}^{k} p_i^{e_i}$ with m ≥ 0, k ≥ 0, 2 < p_1 < … < p_k primes, and $e_i \in \mathbb{N}$, $e_i \geq 1$, for all 1 ≤ i ≤ k, be the prime factorization of n. If $c_n = \sigma_0(q)$ is odd and $w_n = 1$ then for all 1 ≤ i ≤ k, e_i is even, and for any two odd divisors f < g of n, $2^{m+1} \times$ f < g.

PROOF:

If k = 0 then n = 2^m and its symmetric representation has one region of width 1 (see the comments and links in A238443). Let now k > 0, then n must have at least one odd divisor greater than 1. Furthermore, since $c_n = \sigma_0(q) = \prod_{i=1}^{k}(e_i + 1)$ is odd all e_i,1 ≤ i ≤ k, are even, and there is an odd number of 1's in the n-th row of irregular triangle A237048. Since $w_n = 1$ the positions of the odd divisors d_i, 1 ≤ i ≤ $\sigma_0(q)$ = 2×x+1, represented by 1's in the n-th row of irregular triangle A237048 alternate between odd and even positions, i.e.,

$1 = d_1 < 2^{m+1} < d_2 < 2^{m+1} \times d_2 < \ldots < d_x < 2^{m+1} \times d_x < d_{x+1} \leq r_n$.

This chain of inequalities holds for all odd divisors since for

$d_i \times d_{2x+2-i} = d_{i+1} \times d_{2x+1-i} = q$ we get $d_{2x+1-i} < d_{2x+2-i}$ so that

$2^{m+1} \times d_{2x+1-i} = \frac{2^{m+1} \times d_i}{d_{i+1}} \times d_{2x+2-i} < d_{2x+2-i}$.

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THEOREM:

For every number $n \in \mathbb{N}$ with prime factorization $n = 2^m \times q = 2^m \times \prod_{i=1}^{k} p_i^{e_i}$ with $m \geq 0$, $k \geq 0$, $2 < p_1 < \ldots < p_k$ primes, and $e_i \in \mathbb{N}$, $e_i \geq 1$, for all $1 \leq i \leq k$:
c_n is odd & $w_n = 1 \Leftrightarrow n \in$ A241010
$\Leftrightarrow$ for all $1 \leq i \leq k$, e_i is even, and for any two odd divisors $f < g$ of n, $2^{m+1} \times f < g$.

As in the proofs above, let the odd divisors of n be $1 = d_1 < \ldots < d_x < d_{x+1} < \ldots < d_{2x+1} = q$, where $2 \times x+1 = \sigma_0(q)$. The z-th region of n has area $a_z = \frac{1}{2} \times \left(2^{m+1} - 1\right) \times (d_z + d_{2x+2-z})$, for $1 \leq z \leq 2 \times x+1$, so that in this case $v_n = \sum_{z=1}^{2x+1} a_z = \sum_{z=1}^{2x+1} \frac{1}{2} \times \left(2^{m+1} - 1\right) \times (d_z + d_{2x+2-z}) = \left(2^{m+1} - 1\right) \times \left(\sum_{z=1}^{x} (d_z + d_{2x+2-z}) + d_{x+1}\right) = \sigma(n)$.

PROOF:

The equivalences follow from Lemmas A & B. In order to verify the formula for the areas a_z, $1 \leq z \leq 2 \times x+1$, we establish the following identities for the n-th row of irregular triangle E (A235791) that together show $v_n = \sigma(n)$ in this case. Since all regions have width 1, their respective areas are
$\sum_{j=d_z}^{2^{m+1} \times d_z - 1} f_{n,k} = e_{n,d_z} - e_{n,2^{m+1} \times d_z}$, for all $1 \leq z \leq x$, and
$2 \times \sum_{j=d_{x+1}}^{r_n} f_{n,k} - 1 = 2 \times \left(n - \sum_{j=1}^{d_{x+1}-1} f_{n,k}\right) - 1 = 2 \times (n - e_{n,1} - e_{n,d_{x+1}}) - 1 = 2 \times e_{n,d_{x+1}} - 1 = \left(2^{m+1} - 1\right) \times d_{x+1}$,
for the center region a_{x+1} that crosses the diagonal of the Dyck path.

(i) $e_{n,2^{m+1} \times d_z} = e_{n-1,2^{m+1} \times d_z} + 1 = \frac{1}{2} \times \left(\frac{q}{d_z} - 1\right) - 2^m \times d_z + 1$

(ii) $e_{n,d_z} = e_{n-1,d_z} + 1 = 2^m \times \frac{q}{d_z} - \frac{1}{2}(d_z + 1) + 1$

(iii) $e_{n,d_z} - e_{n,2^{m+1} \times d_z} = \frac{1}{2} \times \left(2^{m+1} - 1\right) \times (d_z + d_{2x+2-z})$

(iv) $e_{n,d_{x+1}} = \frac{1}{2} \times \left(2^{m+1} - 1\right) \times d_{x+1} + \frac{1}{2}$

(v) $e_{n,k} = e_{n-1,k}$, for all $1 \leq k \leq r_n$ with $k \neq d_z, 2^{m+1} \times d_z$,

Formulas (i) - (iv) are straightforward calculations. For (v) we argue as follows.
Let $n = u \times k + v$ with $0 \leq v < k$. Then
$e_{n,k} = \left\lceil \frac{u \times k + v + 1}{k} - \frac{k+1}{2} \right\rceil = u + \left\lceil \frac{v+1}{k} - \frac{k+1}{2} \right\rceil$ and $e_{n-1,k} = u + \left\lceil \frac{v}{k} - \frac{k+1}{2} \right\rceil$.
If k is odd and $k \neq d_z$ for any $1 \leq z \leq x$ then $\left\lceil \frac{v+1}{k} \right\rceil = \left\lceil \frac{v}{k} \right\rceil = 1$.
If k is even and $k \neq 2^{m+1} \times d_z$ for any $1 \leq z \leq x$ then
$f_{n,k} = u - \frac{k}{2} + \left\lceil \frac{v+1}{k} - \frac{1}{2} \right\rceil$ and $f_{n-1,k} = u - \frac{k}{2} + \left\lceil \frac{v}{k} - \frac{1}{2} \right\rceil$.
Case $0 \leq v < \frac{k}{2}$:
$\left\lceil \frac{v+1}{k} - \frac{1}{2} \right\rceil = 0 = \left\lceil \frac{v}{k} - \frac{1}{2} \right\rceil$ since $2 \times v < k$ and k even imply $2 \times v + 2 \leq k$.
Case $\frac{k}{2} < v < k$:
$\left\lceil \frac{v+1}{k} - \frac{1}{2} \right\rceil = 1 = \left\lceil \frac{v}{k} - \frac{1}{2} \right\rceil$ since $0 < 2 \times v - k$.
Case $\frac{k}{2} = v$:
In this case $n = u \times k + v = u \times k + \frac{k}{2} = \frac{k}{2} \times (2 \times u + 1)$ so that $2 \times n = 2^{m+1} \times q = (2 \times u + 1) \times k$.
This implies that $2^{m+1} | k$ and $k = 2^{m+1} \times d_z$, for some z, contradicting the assumption on k.
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