Theorem $T(n, A003056(n)) - T(n-1, A003056(n-1)) = \sigma(n)$

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T(n, k) = Sum_{j = 1 .. k} (-1)^(j + 1) * S(n, j)^2, n >= 0, 1 <= k <= A003056(n), where S(n, j) is the j-th entry in the n-th row of the triangle of A235791 .

Case 1 : A003056(n) = A003056(n-1) = m

$$\begin{aligned}
&T(n, A003056(n)) - T(n-1, A003056(n-1)) \\
&= \sum_{k=1}^{m} (-1)^{k+1} (A235791(n,k))^2 - \sum_{k=1}^{m} (-1)^{k+1} (A235791(n-1,k))^2 \\
&= \sum_{k=1}^{m} (-1)^{k+1} \left(\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil\right)^2 - \sum_{k=1}^{m} (-1)^{k+1} \left(\left\lceil \frac{n}{k} - \frac{k+1}{2} \right\rceil\right)^2 \\
&= \sum_{k=1}^{m} (-1)^{k+1} \left(\left(\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil\right)^2 - \left(\left\lceil \frac{n}{k} - \frac{k+1}{2} \right\rceil\right)^2\right) \\
&= \sum_{k=1}^{m} (-1)^{k+1} \left(\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil - \left\lceil \frac{n}{k} - \frac{k+1}{2} \right\rceil\right) \times \left(\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil + \left\lceil \frac{n}{k} - \frac{k+1}{2} \right\rceil\right) \\
&= \sum_{k=1}^{m} (-1)^{k+1} A237048(n,k) \times \left(\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil + \left\lceil \frac{n}{k} - \frac{k+1}{2} \right\rceil\right) =
\end{aligned}$$

now, $\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil = \left\lceil \frac{n}{k} - \frac{k+1}{2} \right\rceil + 1$ precisely when A237048(n, k) = 1, therefore,

$$\begin{aligned}
&= \sum_{k=1}^{m} (-1)^{k+1} A237048(n,k) \times \left(\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil + \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil - 1\right) \\
&= 2 \times \sum_{k=1}^{m} (-1)^{k+1} A237048(n,k) \times \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil - \sum_{k=1}^{m} (-1)^{k+1} A237048(n,k) \\
&= \sigma(n)
\end{aligned}$$

For the last step in the argument see the Comments section in A249223 and the Links sections in A264116 and A280851 proving that the symmetric representation of sigma for n equals sigma(n).

Case 2 : A003056(n) = a003056(n - 1) + 1

In this case n = m(m+1)/2 is a triangular number, A003056(n) = m, S(n, m) = 1, and value S(n-1, m) = 0 extends row n-1 of A235791. With these values the computation of Case 1 applies.

■

As a consequence, the right border of the irregular triangle in A236630 is sequence A024916.