A Convex Lens

NUMBER THEORY

MAHLER'S QUINARY CONUNDRUM

APRIL 28, 2015 | DRADCLIFFE@GMAIL.COM | 1 COMMENT

The following question was posed by the German mathematician Kurt Mahler (1903 – 1988) in a letter that he wrote to Alf van der Poorten.

> I am interested in the problem of whether there are squares of integers which, to the base g = 5, have only digits 0 or 1. I could not find a single example although I went quite far on my calculator.

Now some examples come immediately to mind. If $n = 5^k$ then $n^2 = 5^{2k}$ which is expressed in base 5 as $1000\ldots0$ with $2k$ zeros. But these solutions are trivial. We are more interested in solutions that are not divisible by 5, since they generate all other solutions. If n^2 has only digits 0 or 1, then the same is true for $(5^k \cdot n)^2$ and conversely. (The word "digits" always refers to base 5 digits in this note.)

I wrote a Python script to search for examples. The first step is to write a function to check if a number has only digits 0 or 1 in a given base.

```
def check(n, base):
    while n > 0:
        if n % base > 1:
            return False
        n = n // base
    return True
```

We could use this function to search for examples directly, but this is inefficient.

```
def naive_search(N):
    for n in xrange(N):
```

```
        if n % 5 > 0 and check(n*n, 5):
            yield n
```

It is inefficient because we are checking many values that could have been excluded from the outset. For example, the last digit of *n* must be 1 or 4, otherwise the last digit of n^2 would be 4. We can find similar conditions on the last two digits, the last three digits, and so on.

- The last two digits of *n* must be 01, 14, 31, or 44.
- The last three digits of *n* must be 001, 031, 114, 144, 301, 331, 414, or 444.
- The last four digits of *n* must be 0001, 0301, 0331, 0414, 1444, 2031, 2114, 2144, 2301, 2331, 2414, 3001, 4031, 4114, 4144, or 4444.

In fact, the number of possible k-digit endings is 2^k. Each k-digit ending gives birth to two $(k+1)$-digit endings by adding digits to the left. I will leave it to the reader to prove this fact.

Here is a Python function that generates the possible k-digit endings. It works recursively by generating and extending the possible $(k-1)$-digit endings.

```
def gen(k):
    if k == 1:
        yield 1
        yield 4
    else:
        P = 5 ** (k-1)
        for n in gen(k-1):
            x = (n*n) // P
            delta = 2 * (n % 5) % 5
            d = x % 5
            m = n
            for a in xrange(5):
                if d < 2:
                    yield m
                d = (d + delta) % 5
                m += P
```

The final step is to feed these candidates into our digit checker. By this means, I was able to find all positive integers n less than $5^{32} \approx 2.3 \times 10^{22}$ and not divisible by 5 such that n^2 has only digits 0 or 1 when written in base 5.

```
results = sorted([n for n in gen(32)
                    if check(n*n, 5)])
```

Results: 1, 972799, 3051273374, 6132750376, 839228035909, 3818357814376, 2384643515634376, 1490173338867234376, 931329727148437734376.

If we write these numbers in base 5, a pattern becomes apparent. Here is a Python script that converts the numbers to base 5.

```
def toBase5(n):
    if n < 5:
        return n
    return 10 * toBase5(n//5) + (n % 5)

for n in results:
    print toBase5(n)
```

INDEX	BASE 10	BASE 5
1	1	1
2	972799	222112144
3	3051273374	22222111221444
4	6132750376	100024441003001
5	839228035909	102222214334122114
6	3818357814376	1000244441000030001
7	2384643515634376	1000002444441000003000001
8	1490173338867234376	10000002444444100000030000001
9	931329727148437734376	100000002444444410000000300000001

One sees a similarity between the fourth, sixth, seventh, eighth, and ninth terms of this sequence. The runs of 0s and 4s increase by one in each step. This can be described by a polynomial function. If $P(x) = 25x^4 + 15x^3 - 4x^2 + 3x + 1$then these terms are $P(5^3), P(5^4), P(5^5), P(5^6)$, and $P(5^7)$.

We can verify that

$$P(x)^2 = 625x^8 + 750x^7 + 25x^6 + 30x^5 + 156x^4 + 6x^3 + x^2 + 6x + 1.$$

Note that the coefficients are positive integers less than 5^5 and they can be written in base 5 using only 0s and 1s. Therefore, $(P(5^k))^2$ has only digits 0 or 1 for all $k \geq 3$.

This gives a striking, albeit partial, answer to Mahler's question. There exist infinitely many squares that are not divisible by 5, but which have only the digits 0 or 1 when written in base 5.

Credits: Thanks to Gary Davis (@republicofmath) for bringing this problem to my attention. The problem is mentioned in the article *The Legacy of Kurt Mahler*, which appears in the May 2015 edition of the Notices of the American Mathematical Society.

ONE THOUGHT ON "MAHLER'S QUINARY CONUNDRUM"

★ **dradcliffe@gmail.com**

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This is sequence A230030 in the On-Line Encyclopedia of Integer Sequences.