CL-Chemy III: Hyper-Quadratics

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A Synopsis of the Basics as Covered in CL-Chemy I and II

The idea of the *CL*, or *coefficient list*, is that coefficients may assume diverse values over the course of an iterative procedure. In one preceding article, 'CL-Chemy Transforms Fibonacci-type Sequences to Arrays' (CL-Chemy I), the formula $(c)F_n + (b)F_{n+1} = F_{n+2}\,;\, F_0 = 0,\ F_1 = 1$ is generalized to

$$(\gamma)F_n + (\beta)F_{n+1} = F_{n+2}\ ; \quad F_0 = 0,\ F_1 = 1$$

Where β and γ are the lists $\beta = [b_1, b_2\ldots\, b_i]$ and $\gamma = [c_1, c_2\ldots\, c_j]$.

A sequence φ_λ (where λ = the *order* of φ = LCM(i,j)) is generated by applying terms in β and γ in order, according to the iteration being performed. So, at the 1st iteration, the initial F_0 and F_1 are multiplied by c_1 and b_1 respectively. At the 2nd iteration, F_1 and F_2 are multiplied by c_2 and b_2; on the 3rd iteration c_3 and b_3 apply, and so on. After λ iterations, the cycle repeats.

That generates the first sequence: to start with $\beta = [b_2, b_3\ldots\, b_i, b_1]$ and $\gamma = [c_2, c_3\ldots\, c_i, c_1]$ generates the next. Permuting β and γ cyclically generates λ distinct sequences. In the context of an array, these sequences are aligned vertically and designated as S_1, S_2, S_3 and so forth. Arrays are typically represented by $\Phi_\lambda\,[\beta][\gamma]$, with β, γ and λ in numerical form.

Define F_n/F_{n-1}, for $n \to \infty$, as a *limit ratio*. As a rule, each sequence in an array converges, simultaneously and in two directions, to λ positive and λ negative limit ratios. Formulas derived in CL-Chemy I operate on the elements of Φ_λ to provide the coefficients of λ different quadratic equations (Q_j) that have roots corresponding to specific limit ratios. These sets of equations are called *Q-sets*.

In the article CL-Chemy II: Reflections and Other Symmetries, inverting the order of the terms in β and γ created complementary Q-sets. Further investigation revealed unexpected crossover connections between the roots of equations in these sets. This paper follows up on the discovery that symmetrical inversions of certain β and γ configurations allow roots from the two sets to combine as points on hyperbolic curves.

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Quadratic Root Reflections on Hyperbolic Curves

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Symmetric Inversions

Definition: A *symmetric* inversion $\Phi_\lambda \to \not\Phi_\lambda$ is symmetrical with respect to the order of terms in β and γ. That is, it directly inverts (reverses) the sequential order of the terms in each of the lists β and γ:

$$\beta \to \not\beta = [b_1, b_2 \dots b_\lambda] \to [\not b_\lambda, \not b_{\lambda-1} \dots \not b_1] \qquad \gamma \to \not\gamma = [c_1, c_2 \dots c_\lambda] \to [\not c_\lambda, \not c_{\lambda-1} \dots \not c_1]$$

For example:

	Φ_3 [1,2,3][1,2,3]			$\not\Phi_3$ [3,2,1][3,2,1]		
	S_1	S_2	S_3	S_1	S_2	S_3
F_{-4}	$^{-14}/_{18}$	$^{-15}/_{6}$	$^{-11}/_{12}$	$^{-14}/_{6}$	$^{-11}/_{18}$	$^{-15}/_{12}$
$F_{-\lambda}$	$^{4}/_{6}$	$^{9}/_{6}$	$^{4}/_{6}$	$^{8}/_{6}$	$^{3}/_{6}$	$^{6}/_{6}$
F_{-2}	$^{-2}/_{6}$	$^{-3}/_{3}$	$^{-1}/_{2}$	$^{-2}/_{2}$	$^{-1}/_{3}$	$^{-3}/_{6}$
F_{-1}	$^{1}/_{3}$	$^{1}/_{1}$	$^{1}/_{2}$	$^{1}/_{1}$	$^{1}/_{3}$	$^{1}/_{2}$
F_0	0	0	0	0	0	0
F_1	1	1	1	1	1	1
F_2	1	2	3	3	2	1
F_λ	4	9	4	8	3	6
F_4	15	11	14	11	15	14
F_5	19	40	54	57	36	20
$F_{2\lambda}$	68	153	168	136	51	102

Table 1: Symmetric inversion of Φ_3 [1,2,3][1,2,3]

An equation that uses terms from this array to construct quadratic equations (Q_j) is stated below. The zeros (roots) of these Q_j are the limit ratios to which the quotients F_n/F_{n-1} in table 1 columns converge.

$$Q_j = F_{\lambda,\,j} \cdot x_j^2 - (F_{\lambda+k,\,j} - F_{\lambda-k,\,j+k} \cdot c)\,x_j - F_{\lambda,\,j+k} \cdot c \tag{1.1}$$

Now (1.1) applies to table 1 sequences to create the equations in table 2 (color-coded for future reference). We'll call these two sets of three equations, *Q-sets*. (The order of the last two $\not Q_j$ in the set to the right has been reversed, so as to have Q_j with identical c coefficients on the same line.)

$$Q_1 = 4x_1^2 - 13x_1 - 9 \qquad \not Q_1 = 8x_1^2 - 5x_1 - 9$$
$$Q_2 = 9x_2^2 - 5x_2 - 8 \qquad \not Q_2 = 6x_2^2 - 11x_2 - 8$$
$$Q_3 = 4x_3^2 - 11x_3 - 12 \qquad \not Q_3 = 3x_3^2 - 13x_3 - 12$$

Table 2: Q_j and $\not Q_j$ coefficients as derived from the columns in table 1

Next is a demonstration of how roots of these equations combine as points on the hyperbolic curve in figure 1, below. It is instructive to examine a simple case first.

Consider the equation $Q = ax^2 + bx + c$. Let $a = 1$ and $b = c$. Then two formulas that equate the roots of this equation, r_+ and r_- to its coefficients will combine as below to create a third formula:

i) $r_+ + r_- = -b$ ii) $r_+ \cdot r_- = c$ iii) $r_+ \cdot r_- + r_+ + r_- = 0$

A more familiar rendition of iii is

$$xy + x + y = 0 \tag{1.2}$$

Thus the roots of, say, $Q_\phi = x^2 - x - 1$ will combine to identify two points, in black on the graph of (1.2), in figure 1 below. (Roots equate in turn to both x and y; hence, two points per pair.)

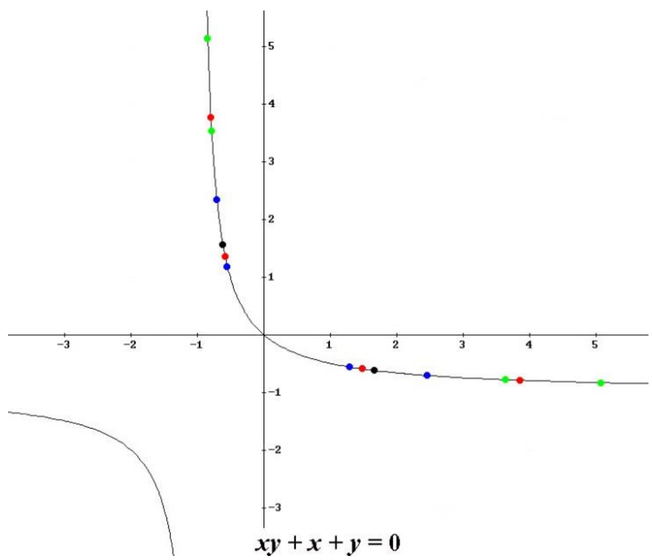

Figure 1: Quadratic roots combine as points on a hyperbola

The graph of (1.1) is hyperbolic, comprising curved lines that are mirror images, symmetrical with respect to two diagonals. One passes through the origin at (0,0) and a perpendicular intersects that at (–1,–1). Call the curved line that transits the origin l_1 and the other l_2. Of the various points that are marked on l_1; those black in color are, again, the roots of Q_ϕ. We'll see now how the others are derived.

When a quadratic with $b = c$ has real roots, points on l_1 or l_2 are identified. For $b \neq c$, it will be seen that, in general, a *pair* of equations is required, and a root from each, taken in combination, identifies a point on the hyperbola. Although $\beta = \gamma$ in table 1, table 2 equations' roots must pair up for a point on (1.2).

For example, take the equations atop table 2, Q_1 and ~~Q~~$_1$. $4x^2 - 13x - 9$ has roots $r_{1+} = 3.8365$, $r_{1-} = -0.5865$; $8x^2 - 5x - 9$ has roots ~~r~~$_{1+} = 1.4182$, ~~r~~$_{1-} = -0.7932$. Then in (1.2), $3.8365(-0.7932) + 3.8365 -0.7932 = 1.4182(-0.5865) + 1.4182 -0.5865 = 0$. These numbers locate the four red dots in figure 1.

A symmetry identified earlier in the graph of (1.1) is now used to map points from l_1 to l_2. Since $x = y$ at both (0,0) and (–2,–2), this entails adding 2 to each root and reversing its sign. Thus to map the black dots to l_2, take $(1.618 + 2)(-1) = -3.618$ and $(-0.618 + 2)(-1) = -1.382$. Multiplying $(x + 3.618)(x + 1.382)$ then gives the coefficients of a new equation: $x^2 + 5x + 5$.

Adapted, this procedure also maps the colored dots to l_2 and gives the equations below.

$$
\begin{array}{ll}
Q_1' = 4x_1^2 + 29x_1 + 33 & \qquad \not{Q}_1' = 8x_1^2 + 37x_1 + 33 \\
Q_2' = 9x_2^2 + 41x_2 + 38 & \qquad \not{Q}_2' = 6x_2^2 + 35x_2 + 38 \\
Q_3' = 4x_3^2 + 27x_3 + 26 & \qquad \not{Q}_3' = 3x_3^2 + 25x_3 + 26
\end{array}
$$

Table 3: Quadratics built on roots reflected to l_2

There are now, in table 3, two more sets of three equations, each with integer coefficients. The roots of these reside, a $+\sqrt{b^2 - 4ac}$ root paired with its complementary $-\sqrt{b^2 - 4ac}$, on the line l_2. Based on the structure of this mapping algorithm, it seems appropriate to identify equations with roots on l_1 as *primary*, and those with roots on l_2 as *secondary* or *reflected*.

Certain attributes of the primary Q_j are unchanged by mapping l_1 roots to l_2; e.g., the a coefficients, their shared discriminant and so forth. Some reasons for this will become clearer later on.

The procedure so far: CLs (coefficient lists) in a generalized Fibonacci sequence formula generate sets of sequences that are aligned in an array. Now, given $\beta = \gamma$ (within certain constraints to be referred to later), symmetrically inverting their terms gives another, closely related array. Then the formula (1.1) applies to find coefficients of equations that have roots to which the ratios of adjacent terms in the sequences converge. These equation form two sets, where Q_j roots combine with roots of a $\not{Q}_i$ as points on a hyperbolic line.

Then, as just observed, the roots on l_1 can be mapped to l_2, and coefficients for two new sets of equations derived from that. So, in sum, these procedures require us to: 1) construct the arrays Φ_λ and $\not{\Phi}_\lambda$; 2) derive the coefficients for the primary equations Q_j and $\not{Q}_i$; 3) map those roots to l_2, and then 4) find coefficients for two sets of secondary equations from these roots. Can a more direct method be found?

Quadratic Coefficient Matrices

Remarkably, a shortcut exists that obviates all of this, save construction of one array and a solitary Q-set. The Q_j coefficients are configured as a matrix, and a set of transforms finds the related sets from that.

To derive the first transform matrix, let coefficients of table 2 equations be arrayed in two 3x3 matrices M and $\not{M}$ as below. The operation $M^{-1} \cdot \not{M} = \not{M}^{-1} \cdot M$ produces, as a 'quotient', a matrix (T_1) that transforms one to the other: i.e., $M \cdot T_1 = \not{M}$ and $\not{M} \cdot T_1 = M$.

$$
\begin{pmatrix} 4 & -13 & -9 \\ 9 & -5 & -8 \\ 4 & -11 & -12 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 8 & -5 & -9 \\ 6 & -11 & -8 \\ 3 & -13 & -12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = T_1
$$

For the Q_j' with roots that identify points on l_2, matrix M' is fashioned from the coefficients at the left side of table 3. Then $M^{-1} \cdot M' = T_2$, the transform matrix at the right below:

$$
\begin{pmatrix} 4 & -13 & -9 \\ 9 & -5 & -8 \\ 4 & -11 & -12 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 & 29 & 33 \\ 9 & 41 & 38 \\ 4 & 27 & 26 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = T_2
$$

Finally, $M^{-1} \cdot M'$ finds the last of the four transforms to complete the group below:

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & 4 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 4 & 4 \\ -1 & -3 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$

These T-matrices are isomorphic by matrix multiplication to the Klein four-group. Now, in theory, we need to construct (according to conditions on β and γ) but one array to find all of the related Q-set coefficients. Since an M created on Q_λ coefficients has three columns, and T_n has three rows, they are *conformal* and thus T_n works for all $\lambda \geq 1$. For two such examples, take matrices built on coefficients of the Φ_2 [1,2][1,2] and Φ_4 [1,2,3,4][1,2,3,4] Q-sets, and apply the transforms.

$$M = \begin{pmatrix} 1 & -3 & -2 \\ 2 & -1 & -2 \end{pmatrix} \quad M \cdot T_1 = \begin{pmatrix} 2 & -1 & -2 \\ 1 & -3 & -2 \end{pmatrix} \quad M \cdot T_2 = \begin{pmatrix} 1 & 7 & 8 \\ 2 & 9 & 8 \end{pmatrix} \quad M \cdot T_3 = \begin{pmatrix} 2 & 9 & 8 \\ 1 & 7 & 8 \end{pmatrix}$$

$$M = \begin{pmatrix} 44 & -21 & -38 \\ 19 & -55 & -54 \\ 18 & -53 & -60 \\ 15 & -67 & -44 \end{pmatrix} \quad M \cdot T_1 = \begin{pmatrix} 27 & -55 & -38 \\ 20 & -53 & -54 \\ 11 & -67 & -60 \\ 38 & -21 & -44 \end{pmatrix} \quad M \cdot T_2 = \begin{pmatrix} 44 & 197 & 180 \\ 19 & 131 & 132 \\ 18 & 125 & 118 \\ 15 & 127 & 150 \end{pmatrix} \quad M \cdot T_3 = \begin{pmatrix} 27 & 164 & 180 \\ 20 & 133 & 132 \\ 11 & 111 & 118 \\ 38 & 173 & 150 \end{pmatrix}$$

Let's try a sample pair to be sure we're on track: the r_+ root of $18x^2 + 125x + 118$ is –1.1268… and $11x^2 + 111x + 118$ has an r_- root of –8.8833… These numbers zero out in (1.2) and identify two points on l_2.

These transform matrices simplify things considerably; yet they properly apply only to a specific case (i.e., $k = 1$) of the more general version of (1.2) below:

$$kxy + x + y = 0 \tag{1.3}$$

For an example, let $k = 2$: then matrices constructed on Φ_3 $[2c_1,2c_2,2c_3][c_1,c_2,c_3]$ coefficients have roots on $2xy + x + y$. The transforms (T') that work for these matrices are juxtaposed above the originals below.

$$T_0' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_1' = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 4 & 4 & 1 \end{pmatrix} \quad T_2' = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad T_3' = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & -1 \\ 4 & 4 & 1 \end{pmatrix}$$

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & 4 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 4 & 4 \\ -1 & -3 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$

Strangely, the T-matrices themselves seem to have undergone transformations, where 1 and 2 have swapped places and all three are rotated by 180º. Generalizing these transforms clears this up. We'll see that T_1 and T_2 have an inverse relationship. One shrinks as the other grows and, together, these matrices model both the curvature of $kxy + x + y$ and the distance between its lines. (I.e., note that the absolute value of both x and y at the apex of l_2 is $2/k$.) Setting k as an upper index, the set of generalized T-matrices that transform a matrix built on coefficients in the set Q_λ derives, heuristically, as:

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_1^k = \begin{pmatrix} 1 & 0 & 0 \\ -k & -1 & 0 \\ k^2 & 2k & 1 \end{pmatrix} \qquad T_2^k = \begin{pmatrix} 1 & 4/k & 4/k^2 \\ 0 & -1 & -2/k \\ 0 & 0 & 1 \end{pmatrix} \qquad T_3^k = \begin{pmatrix} 1 & 4/k & 4/k^2 \\ -k & -3 & -2/k \\ k^2 & 2k & 1 \end{pmatrix}$$

Table 4: The generalized transformation matrix set

As a check, these table 4 T-matrices too are isomorphic to the Klein group.

Another Q-set example uses M composed of coefficients of $\lambda = 3$ equations with roots on $3xy + x + y = 0$.

$$M = \begin{pmatrix} 20 & -183 & -57 \\ 57 & -159 & -56 \\ 28 & -177 & -60 \end{pmatrix} \qquad M \cdot \begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & 0 \\ 9 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 56 & -159 & -57 \\ 30 & -177 & -56 \\ 19 & -183 & -60 \end{pmatrix} = \not{M}$$

$$M \cdot \begin{pmatrix} 1 & 4/3 & 4/9 \\ 0 & -1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 180 & 1887 & 665 \\ 513 & 2115 & 678 \\ 252 & 1929 & 634 \end{pmatrix} = M' \qquad M \cdot \begin{pmatrix} 1 & 4/3 & 4/9 \\ -3 & -3 & -2/3 \\ 9 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 504 & 2103 & 665 \\ 270 & 1953 & 678 \\ 171 & 1875 & 634 \end{pmatrix} = \not{M}'$$

Table 5:

Note that clearing the fractions makes for larger coefficients in M' and $\not{M}'$. As a check on the transforms, solve $180x^2 + 1887x + 665$ for roots –0.3651 and –10.1182, and $504x^2 + 2103x + 665$ for the roots –0.3447 and –3.8279. Then $3(-10.1182)(-0.3447) - 10.1182 - 0.3447 = 0$.

The Mechanics of the Transform Matrices

Yet it turns out that arrays, Q-sets and all that don't matter: a T-matrix works on *any* 3-column matrix M (but zeros in M can give strange results). Random matrices are generated below to see what comes of it…

$$\begin{pmatrix} 12 & 14 & 13 \\ 2 & -1 & -6 \\ 4 & 24 & -13 \\ 15 & 23 & -19 \\ -22 & 14 & 17 \end{pmatrix} \cdot T_1 = \begin{pmatrix} 11 & 12 & 13 \\ -3 & -11 & -6 \\ -33 & -50 & -13 \\ -27 & -61 & -19 \\ -19 & 20 & 17 \end{pmatrix} \qquad \begin{pmatrix} 8 & 7 & -3 \\ -8 & -4 & -25 \\ 2 & -13 & 15 \\ 21 & -15 & -7 \\ -22 & -13 & -23 \end{pmatrix} \cdot T_1 = \begin{pmatrix} -2 & -13 & -3 \\ -29 & -46 & -25 \\ 30 & 43 & 15 \\ 29 & 1 & -7 \\ -32 & -33 & -23 \end{pmatrix}$$

Table 6: Two 5x3 random matrices transformed by T_1

We encounter something different at once; i.e., the top equation $12x^2 + 14x + 13$ has (conjugate) complex roots; $r_+ = -0.5417 + 0.9345i$ and $r_- = -0.5417 - 0.9345i$. As expected, roots of $11x^2 + 12x + 13$ are complex and conjugate as well; $\not{r}_+ = -0.5833 + 0.8620i$ and $\not{r}_- = -0.5833 - 0.8620i$. These imaginary components notwithstanding, the complementary roots combine as usual as solutions to $xy + x + y = 0$. But how are pairs such as these to be graphed as points on a line or surface?

The next example is also unusual, in that the roots of $2x^2 - x - 6$ combine with $-3x^2 - 11x - 6$ roots in two ways. That is, the roots 2 and –0.6666 identify a point on l_1, but –3 and –1.5 are points on l_2. The roots of equations that T_2 and T_3 return have the same pattern.

Other variants likely await in these and other random examples, but rather than work more table 6 pairings, let's examine the mechanics of matrix multiplication to try to fathom the why of it. E.g., take the equation $ax^2 + bx + c$. Part of the mystery vanishes when we see $(a\ b\ c) \cdot T_1$ multiplied out.

$$\begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} a(1)+b(-1)+c(1) & b(-1)+c(2) & c(1) \end{pmatrix} = \begin{pmatrix} a-b+c & 2c-b & c \end{pmatrix}$$

Figure 2: The mechanics of the T_1 transform

Figure 2 illustrates the T_1 mechanism, but it is still astonishing that this simple shuffle of a Q's coefficients creates a ~~*Q*~~ with these peculiar complimentary properties inherent to its roots. For $k = 1$, T_1 is now passé, superfluous, old news: just put *a*, *b* and *c* into the ~~*a*~~ $= a - b + c$, ~~*b*~~ $= 2c - b$, ~~*c*~~ $= c$ formulas and we're done.

For the general equation set, let $M = (a\ b\ c)$ and multiply it in turn by the T_n^k in table 4.

$$i): \quad M \cdot T_1^k = \begin{pmatrix} ck^2 - bk + a & 2ck - b & c \end{pmatrix} \qquad ii): \quad M \cdot T_2^k = \begin{pmatrix} a & 4a/k - b & 4a/k^2 + 2b/k + c \end{pmatrix}$$

$$iii): \quad M \cdot T_3^k = \begin{pmatrix} ck^2 - bk + a & 4a/k - 3b + 2ck & 4a/k^2 + 2b/k + c \end{pmatrix}$$

Figure 3: Coefficients for equations with roots as points on $kxy + x + y = 0$

Given any standard quadratic Q, the coefficients in figure 3 find roots that combine and reflect to zero out in $kxy + x + y = 0$ (1.3). If Q's roots are real, the +/– complements combine as points on (1.3)'s lines.

To review, arrays such as found in table 1, or sets of equations such as those in tables 2 and 3, are no longer needed to generate these particular root pairings. We can now choose any combination of real, imaginary or complex numbers for *a*, *b* and *c*, and drop them directly into figure 3 formulas.

While our carefully-crafted Q-sets and T-matrices are no longer requisite for finding root-pairs and points, they were nonetheless vital as a way into this exploration. As we seek our bearings in this newfound, infinite sea of interconnections, perhaps they may still serve to provide a certain order to the inquiry. Since Q-set roots are so closely linked, maybe their relationships will translate to unique geometric patterns in the plane of the hyperbola. At present, they are as waypoints on a near-empty chart, islands and reefs in an endless ocean of random choices. It remains to be seen what other landfalls will loom on this course as this odyssey continues…

Going forward, the scope of this venture expands as rotations and reflections of $xy + x + y = 0$ extend these procedures into other areas of the plane.

Some Symmetries of $xy + bx + cy = 0$

The patterns next to be considered are brought into evidence as the equation in (1.2) is generalized to:

$$xy + bx + cy = 0 \tag{1.4}$$

In the interests of symmetry and simplicity, b and c values are allowed to be either of ±1. Hence, there are four variants of (1.4) to consider:

$$xy + x + y \qquad xy - x - y \qquad xy - x + y \qquad xy + x - y$$

These equations correlate by color to the hyperbolas H_1, H_2, H_3 and H_4 in the graph below:

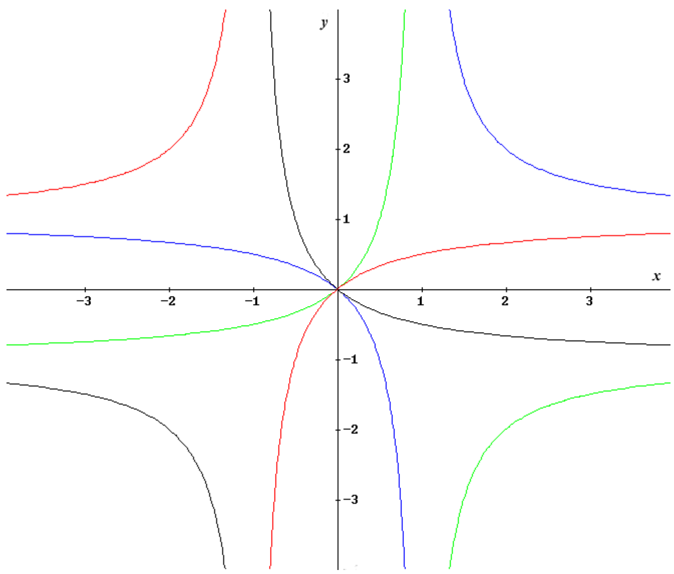

Figure 4: Reflections and rotations of $xy + x + y$

To generate points on H_2, let k take negative values in T_n^k matrices and/or coefficient formulas in figure 3.

To find T-matrices that translate H_1 points to H_3, and H_4 (and to generalize figure 3 formulas to map these root pairs to all of the figure 4 lines) we again use the matrix M formed on the coefficients of the Q-set on the left in table 2:

$$M = \begin{pmatrix} 4 & -13 & -9 \\ 9 & -5 & -8 \\ 4 & -8 & -12 \end{pmatrix}$$

The equations in M are now solved and the roots fed into $xy + x - y$. E.g., $4x^2 - 13x - 9$, has roots $r_+ = 3.8365$ and $r_- = -0.5865$. Then $r_+ \cdot r_-' + r_+ - r_-' = 0$ gives $r_-' = -1.3526$ and similarly for $r_+' = -0.3697$.

Then $(x + r_+')(x + r_-')$ = $18x^2 + 31x + 9$ and we have the top line of a new coefficient matrix. To continue this process with the next two rows of M gives the matrix M' below.

$$M' = \begin{pmatrix} 18 & 31 & 9 \\ 4 & 21 & 8 \\ 19 & 35 & 12 \end{pmatrix}$$

A transform matrix T_4 is found by $M^{-1} \cdot M'$. Next, $r_+ \cdot r_-' - r_+ + r_-' = 0$ and $r_- \cdot r_+' - r_- + r_+' = 0$ could be solved on M's roots to find a second set of roots This process, repeated, gives three rows of M'' and the operation $M^{-1} \cdot M''$ finds a second transform matrix, T_5. But to derive T_5 by use of $M'^{-1} \cdot M$ is much easier. The two new transforms are on the left below, and generalized on the right simply by applying these signage patterns to the original T_1^k.

$$T_4 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \end{pmatrix} \qquad T_4^k = \begin{pmatrix} -1 & 0 & 0 \\ -k & -1 & 0 \\ -k^2 & -2k & -1 \end{pmatrix}, \quad T_5^k = \begin{pmatrix} -1 & 0 & 0 \\ k & -1 & 0 \\ -k^2 & 2k & -1 \end{pmatrix}$$

Multiplying ($a\ b\ c$) by the transforms gives the coefficient sets *iv*) = $(-ck^2 - bk - a \quad -2ck - b \quad -c)$ and *v*) = $(-ck^2 + bk - a \quad 2ck - b \quad -c)$. We'll see later that $k < 0$ reverses the signs of the $k > 0$ roots, and so further reflects points on H_3, and H_4. With these six transforms, is the set for $kxy \pm x \pm y$ complete?

A Distinguishing Property of Transforms Defined

One thing that all equations that are products of these transforms have in common is the discriminant (*D*). A potentially interesting avenue of research tries to identify all possible operations on and/or combinations of ($a\ b\ c$) such that the roots of equations created by transforms will have the same discriminant as the initial $ax^2 + bx + c$. For a look into that…

Note that certain symmetries of T_1 compose the elements of the dihedral group D_3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

Table 7: The dihedral group D_3 represented as reflections and a composition of reflections of T_1

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -k & -1 & 0 \\ k^2 & 2k & 1 \end{pmatrix} \begin{pmatrix} 1 & 2k & k^2 \\ 0 & -1 & -k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k^2 & 2k & 1 \\ -k & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -k \\ 1 & 2k & k^2 \end{pmatrix}$$

Table 8: Elements of table 7 generalized as reflections and a composition of reflections of T_1^k

Here is another set of six transforms of some kind. Is there a geometrical object, analogous to $kxy \pm x \pm y$, upon which the roots generated by these transforms will lie? For $k > 1$, table 8 matrices don't always interact as members of a group, i.e., multiplication often creates new forms. But, again, in numerous trials with diverse, even complex arguments, *D* is invariant. That is, the discriminant of the original equation upon which the transforms work never grows or otherwise changes.

The focus now is on a property that all transform matrices identified so far have in common: they conserve D. To sharpen the focus, let a *transform matrix* (*T-matrix* or *transform*) be defined by the property that it changes a quadratic equation's coefficients in a way that D itself is immutable. A question then is, how/why does this happen? To assist this inquiry, the transforms are represented in the form of quadratic equations below. The first six equations are created by T-matrices of the original set, and the next are from the dihedral group transforms.

$$T_0 \to ax^2 + bx + c$$
$$T_1^k \to (ck^2 - bk + a)x^2 + (2ck - b)x + c$$
$$T_2^k \to ax^2 + (4a/k - b)x + 4a/k^2 + 2b/k + c$$
$$T_3^k \to (ck^2 - bk + a)x^2 + (4a/k - 3b + 2ck)x + 4a/k^2 + 2b/k + c$$
$$T_4^k \to (-ck^2 - bk - a)x^2 + (-2ck - b)x - c$$
$$T_5^k \to (-ck^2 + bk - a)x^2 + (2ck - b)x - c$$

$$T_0 \to ax^2 + bx + c$$
$$T_R^k \to cx^2 + bx + a$$
$$T_1^k \to (ck^2 - bk + a)x^2 + (2ck - b)x + c$$
$$T_{1R}^k \to ax^2 + (2ak - b)x + ak^2 - bk + c$$
$$T_6^k \to (\mathrm{a}k^2 - bk + c)x^2 + (2ak - b)x + a$$
$$T_7^k \to cx^2 + (2ck - b)x - ck^2 - bk + a$$

These 12 examples are a set of 10 unique transforms and equations. The 10 matrices can be paired (allowing repetition) in 55 ways. In trials, some matrix multiplications create a new matrix, and taking some to higher powers. e.g., $(T_4^k)^n$, generates new examples as well. Can it be proved that such products/powers are always another T-matrix? Other efforts to enlarge the inventory, such as rotating T_1 in 90° increments have failed. Among the obvious questions that arise here is: in theory, how many such transforms can exist? If the powers of a T-matrix are taken as separate entities, then there are infinitely many examples, but classification systems could narrow it down to a number of types. If what follows doesn't offer any deep insights into this question, it at least affords a somewhat different approach.

Recall that D of $ax^2 + bx + c$ is $b^2 - 4ac$. With transforms expressed as quadratic equations, coefficients can be inserted into the quadratic formula. E.g., T_1^k gives $a = (ck^2 - bk + a), b = (2ck - b), c = c.$ Working 'by hand', the formula puts $(2ck - b)^2 - 4(ck^2 - bk + a)c$ under the radical. Expanding and collecting terms returns $b^2 - 4ac$.

In computer trials, as these typically bulky coefficients are inserted into quadratic equations, roots returned are sometimes so complicated that it takes a bit of searching just to find the radical. But there, tucked under it, naked and nonchalant, is $b^2 - 4ac$…

Let the size of the set of transforms question be restated in terms of Q's coefficients: given a, b and c, what arrangements, combinations, permutations and sign changes will leave D unaffected? Is this a broader version of the original question; i.e., are there any such forms that a T-matrix won't produce?

Epilog

As this excursion winds to a close, a rich history informs the journey. The story of efforts, circa 1500, to find the roots of polynomials in terms of their coefficients is well known. Inspired initially by the quadratic formula, this quest led, amid other discoveries, to the theory of groups. After all these centuries, it's amazing to see another way in which quadratic coefficients, roots and groups are so closely connected. Given what came of that earlier investigation, the potential for seminal discoveries in this area looks strong…

Problems:

- Provide formal proofs of the important steps in the foregoing procedures.
- Given complex values for x, y and/or k, find graphic representations of $kxy + x + y = 0$.
- Letting k vary continuously, animate the motion of points such as those in figure 1.
- Describe the conditions required on β and γ to ensure that equations taken from arrays created by symmetric inversions have a common discriminant.
- Identify all possible transforms on quadratic coefficients that conserve the discriminant, and prove that the list is complete.
- Generalize these reflective functions to work for polynomials of higher degree.
- Identify any unique relationships that exist between roots generated by the D_3 transform group.

Addenda

It seems likely that a few OEIS candidate sequences can be found around here… one approach is to apply T_2^k, k = 1,2,3…, to the coefficients of $x^2 - x - 1$, and clear the fractions. Table 9 shows results for k = 1..20.

k =	a	b	c	$a+b+c$
20	100	120	–89	131
19	361	437	–319	479
18	81	99	–71	109
17	289	357	–251	395
16	64	80	–55	89
15	225	285	–191	319
14	49	63	–41	71
13	169	221	–139	251
12	36	48	–29	55
11	121	165	–95	191
10	25	35	–19	41
9	81	117	–59	139
8	16	24	–11	29
7	49	77	–31	95
6	9	15	–5	19
5	25	45	–11	59
4	4	8	–1	11
3	9	21	1	31
2	1	3	1	5
1	1	5	5	11

Table 9: T_2^k applied to the coefficients of $x^2 - x - 1$

The sequences in the a and b columns are already listed. The a column terms are A168077. (Note that this sequence is also generated as the square of a_n in A026741; we'll encounter this latter sequence again soon.) The b column in table 9 is the sequence A171621; dividing every 4th term by 4 gives A061037.

The c column's 5,1,1,–1,–11,–5,–31… are now entered in the OEIS as A229526. Note that the sum of a, b and c in a row n is the c coefficient in row n + 4 with the sign reversed (A229525).

k =	a	b	c	$a+b+c$
$^1/_{13}$	13	689	8777	9479
$^1/_{12}$	6	294	3451	3751
$^1/_{11}$	11	495	5315	5821
$^1/_{10}$	5	205	1996	2206
$^1/_9$	9	333	2909	3251
$^1/_8$	4	132	1021	1157
$^1/_7$	7	203	1367	1577
$^1/_6$	3	75	430	508
$^1/_5$	5	105	497	607
$^1/_4$	2	34	127	163
$^1/_3$	3	39	107	149
$^1/_2$	1	9	16	25
1	1	5	5	11

Table 10: T_2^k applied to $x^2 - x - 1$ coefficients with fractional arguments for k

T_2^k applies again in table 10, where *k* takes fractional values; $1, {}^1/_2, {}^1/_3 \ldots {}^1/_n$. The initial coefficients are once more (1 –1 –1); fractions in the coefficients are cleared:

The *a* coefficients in table 10 (A026741) are the square roots of *a* coefficients in table 9. It's interesting that fractional arguments in *k* should have that effect. Such other mathematical relationships as may exist between the coefficients in these two tables are not so obvious…

For T_1^k and (1 –1 –1), $k = 1, {}^1/_2, {}^1/_3 \ldots {}^1/_n$ gives a series of (cleared) coefficients where the sum of terms in row *n* = the *a* coefficient in row *n* – 1 (A002061). Perhaps reference to the sum of *i*, *ii* and *iii* in figure 3 will show why this happens. Given the same *k* and (1 –2 –3), T_2^k generates cleared coefficient sets that sum to –2,–13,–32,–59,–94…, the terms in A185950 with the signs reversed. And so on…

Since we're at it, note the symmetries of T_4^k and T_5^k, the last two transforms from the first set.

	T_4^k				T_5^k			
a+b+c	*a*	*b*	*c*	*k* =	*a*	*b*	*c*	*a+b+c*
55	41	13	1	6	29	–11	1	19
41	29	11	1	5	19	–9	1	11
29	19	9	1	4	11	–7	1	5
19	11	7	1	3	5	–5	1	1
11	5	5	1	2	1	–3	1	–1
5	1	3	1	1	–1	–1	1	–1
1	–1	1	1	0	–1	1	1	1
–1	–1	–1	1	–1	1	3	1	5
–1	1	–3	1	–2	5	5	1	11
1	5	–5	1	–3	11	7	1	19
5	11	–7	1	–4	19	9	1	29
11	19	–9	1	–5	29	11	1	41
19	29	–11	1	–6	41	13	1	55

Table 11: T_4^k and T_5^k apply to (1,– 1,– 1) with *k* = –6..6

References:

N.J.A. Sloane's The Online Encyclopedia of Integer Sequences
http://oeis.org/