Asymptotic of subsequences of A212382

(*Václav Kotěšovec, published July 17 2014*)

In the OEIS (On-Line Encyclopedia of Integer Sequences) published Alois P. Heinz in 2012 sequences "Number of Dyck n-paths all of whose ascents have lengths equal to 1 (mod p)", which can be generalized (for $p \geq 1$) as the family of the sequences with an ordinary generating function A(x), satisfies functional equation

$$A(x) = 1 + x * \frac{A(x)}{1 - \left(x * A(x)\right)^p}$$

Sequences in the OEIS:
A000108 (p=1), A101785 (p=2), A212383 (p=3), A212384 (p=4), A212385 (p=5), A212386 (p=6), A212387 (p=7), A212388 (p=8), A212389 (p=9), A212390 (p=10), all sequences in one array together A212382.

Theorem (V. Kotěšovec, July 16 2014):
The asymptotic is (for $p \geq 1$)

$$a_n \sim \frac{s^2}{n^{3/2}\, r^{n-1/2} \sqrt{2\pi p\,(s-1)\left(1 + \frac{s}{1+p\,(s-1)}\right)}}$$

where r $(0 < r < 1)$ and s are real roots of the system of equations

$$r = \frac{p\,(s-1)^2}{s\,(1-p+ps)} \qquad\qquad (rs)^p = \frac{s-1-rs}{s-1}$$

Proof:

Following theorem by Edward A. Bender is (in case of implicit functions) very useful (for proof see [1], p.505 and also [4], p.469).

Citation: Edward A. Bender, "Asymptotic methods in enumeration" (1974), p.502, see [1]

THEOREM 5. *Assume that the power series* $w(z) = \sum a_n z^n$ *with nonnegative coefficients satisfies* $F(z, w) \equiv 0$. *Suppose there exist real numbers* $r > 0$ *and* $s > a_0$ *such that*

(i) *for some* $\delta > 0$, $F(z, w)$ *is analytic whenever* $|z| < r + \delta$ *and* $|q| < s + \delta$;
(ii) $F(r, s) = F_w(r, s) = 0$;
(iii) $F_z(r, s) \neq 0$, *and* $F_{ww}(r, s) \neq 0$: *and*
(iv) *if* $|z| \leqq r, |w| \leqq s$, *and* $F(z, w) = F_w(z, w) = 0$, *then* $z = r$ *and* $w = s$.

Then

(7.1) $$a_n \sim ((rF_z)/(2\pi F_{ww}))^{1/2} n^{-3/2} r^{-n},$$

where the partial derivatives F_z *and* F_{ww} *are evaluated at* $z = r$, $w = s$.

Bender's formula applied for ordinary generating function is

$$a_n \sim \frac{1}{n\,r^n} \sqrt{\frac{r\,F_z}{2\pi n\,F_{ww}}}$$

(for exponential generating function see [8])

Now we have the implicit function

$$f(x,y) = \frac{xy}{1-(xy)^p} - y + 1$$

partial derivatives		
F_z	$\frac{\partial}{\partial x} f(x,y)$	$\frac{y(p(xy)^p - (xy)^p + 1)}{((xy)^p - 1)^2}$
F_w	$\frac{\partial}{\partial y} f(x,y)$	$\frac{x(p(xy)^p - (xy)^p + 1)}{((xy)^p - 1)^2} - 1$
F_{ww}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$-\frac{px(xy)^p(p(xy)^p - (xy)^p + p + 1)}{y((xy)^p - 1)^3}$

r, s, are roots of the system of equations

$$s\left(\frac{r}{1-(rs)^p} - 1\right) + 1 = 0 \qquad \frac{r\,(p\,(rs)^p - (rs)^p + 1)}{((rs)^p - 1)^2} = 1$$

which can be simplified as

$$r = \frac{p\,(s-1)^2}{s\,(1-p+ps)} \qquad (rs)^p = \frac{s-1-rs}{s-1}$$

Note that s is the root of a polynomial of degree $2p$

$$p^p(s-1)^{2p} - (ps-p+1)^{p-1} = 0$$

The asymptotic is then

$$a_n \sim \frac{r^{\frac{1}{2}-n}\sqrt{-\frac{s^3(rs)^{-p-1}((rs)^p - 1)(p(rs)^p - (rs)^p + 1)}{p(p(rs)^p - (rs)^p + p + 1)}}}{\sqrt{2\pi}n^{3/2}}$$

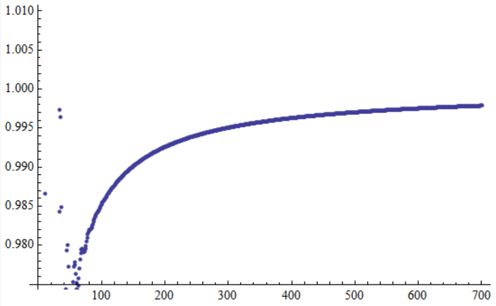
after simplification

$$a_n \sim \frac{s^2}{n^{3/2}\, r^{n-1/2}\,\sqrt{2\pi p\,(s-1)\left(1+\frac{s}{1+p\,(s-1)}\right)}}$$

Numerical verification (for p=6, A212386)

```
N[Solve[{r == p * (s - 1) ^2 / (s * (1 - p + p * s)), (r * s) ^p == (s - 1 - r * s) / (s - 1), r > 0, r < 1} /.
   p → 6, {r, s}, Reals], 20]
{{r → 0.39251327125804462442, s → 1.8766537866430581017}}
```

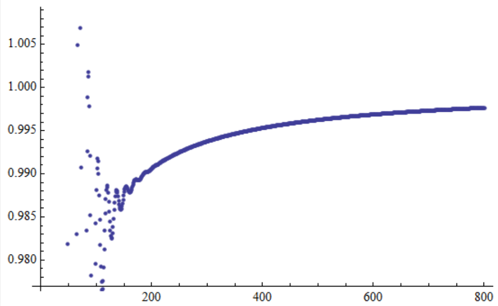
```
ListPlot[
 Table[
  A212386[[n]] /
    (s^2 / (n^(3/2) * r^(n - 1/2) * Sqrt[2 * Pi * p * (s - 1) * (1 + s / (1 + p * (s - 1)))])) /.
   {p → 6, r → 0.3925132712580446244, s → 1.876653786643058101}, {n, 1, Length[A212386]}]]
```



Numerical verification (for p=9, A212389)

```
N[Solve[{r == p * (s - 1) ^2 / (s * (1 - p + p * s)), (r * s) ^p == (s - 1 - r * s) / (s - 1), r > 0, r < 1} /. p → 9,
   {r, s}, Reals], 20]
{{r → 0.41640395155141206718, s → 1.8826164234357634660}}
```

```
ListPlot[
 Table[
  A212389[[n]] /
    (s^2 / (n^(3/2) * r^(n - 1/2) * Sqrt[2 * Pi * p * (s - 1) * (1 + s / (1 + p * (s - 1)))])) /.
   {p → 9, r → 0.4164039515514120671, s → 1.882616423435763466}, {n, 1, Length[A212389]}]]
```



References:

[1] Edward A. Bender, Asymptotic methods in enumeration, SIAM Review 16 (1974), no. 4, 485-515
[2] Kotěšovec V., Asymptotic of implicit functions if Fww = 0, extension of theorem by Bender, website 19.1.2014
[3] OEIS - The On-Line Encyclopedia of Integer Sequences
[4] P. Flajolet and R. Sedgewick, Analytic Combinatorics, 2009, p. 469
[5] Kotěšovec V., Interesting asymptotic formulas for binomial sums, website 9.6.2013
[6] Kotěšovec V., Asymptotic of a sums of powers of binomial coefficients * x^k, website 20.9.2012
[7] Kotěšovec V., Asymptotic of sequences A244820, A244821 and A244822, website (and OEIS) 11.7.2014
[8] Kotěšovec V., Asymptotic of sequences A161630, A212722, A212917 and A245265, website (and OEIS) 16.7.2014

This article was published on the website http://web.telecom.cz/vaclav.kotesovec/math.htm, 17.7.2014
and in the OEIS, July 17 2014