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A196837: Ordinary Generating Functions for Sums of Powers of the First n Positive Integers

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The sum of the $k-$th power of the first n positive integers (we use $S_k(n)$ for the normalized sum),

$$\Sigma\, n^k \;\equiv\; n\, S_k(n) \;:=\; \sum_{j=1}^{n} j^k,\;\; n\,\in\,\mathbb{N},\, k\,\in\,\mathbb{N}_0, \tag{1}$$

has an obvious exponential generating function (*e.g.f.*) $g(n,x) \;:=\; \sum_{k=0}^{\infty} n\, S_k(n)\, \frac{x^k}{k!}$, *viz*

$$g(n,x) \;=\; \sum_{j=1}^{n} e^{j\,x} \;=\; e^x\, \frac{e^{n\,x}\;-\;1}{e^x\;-\;1}\;. \tag{2}$$

The second equation uses the finite geometric sum formula. For given n the sequence $\{\Sigma\, n^k\}_{k=0}^{\infty}$ appears as column no. n in the array [6] A103438 which is there called $T(m,n)$ (this is not a triangle, and this entry uses a $n=0$ column consisting only of zeros because there $0^0\;:=\;0$). See the example array given there.

In order to derive the ordinary generating function (*o.g.f.*) one uses the general connection between an *e.g.f.* $g(x)$ and the corresponding *o.g.f.* $G(x)$, namely $\mathcal{L}[g(t)] \;=\; F(p) \;=\; \frac{1}{p}\, G\left(\frac{1}{p}\right)$, with the *Laplace* transformation $\mathcal{L}$. Thus $G(x) \;=\; \frac{1}{x}\, F\left(\frac{1}{x}\right)$. This connection derives from the elementary *Laplace transform* of the exponential function: $\mathcal{L}[e^{s\,t}] \;=\; \frac{1}{p-s}$. From this the *o.g.f.* corresponding to the *e.g.f.* $e^{s\,t}$ becomes $G(x) \;=\; \frac{1}{1-s\,p}$. Therefore, whenever the *Laplace* transform of an *e.g.f.* $g(x)$ is known, one knows the *o.g.f.* , and *vice versa*.

In the case at hand we thus obtain the *o.g.f.* $G(n,x)$ from the *e.g.f.* $g(n,x)$ (using the linearity of $\mathcal{L}$)

$$G(n,x) \;=\; \sum_{j=1}^{n} \frac{1}{1-j\,x}\,. \tag{3}$$

This is rewritten as

$$G(n,x) \;=\; \frac{P(n,x)}{\prod_{j=1}^{n}\,(1-j\,x)}\,. \tag{4}$$

The numerator polynomials $P(n,x)$ are the row polynomials of this triangle A196837. We list these polynomials (computed by Maple 13 [5]) for $n\;=\;1..15$ in *Table 1*. Sometimes there occurs factorization. Here the partial fraction decomposition (*p.f.d.*) has been performed backwards: one searches for the rational function $G(n,x)$ with a given simple *p.f.d.*. This must have appeared earlier in the literature, but the author was not (yet) able to find it in some standard books.

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An equivalent definition of these row polynomials is thus

$$P(n,x) \;=\; \sum_{l=1}^{n} \frac{1}{1-l\,x}\,\prod_{j=1}^{n}(1-j\,x)\,. \tag{5}$$

It is clear that $P(n,x)$ is a polynomial of degree $n-1$ (see also the later eq. 8). Note, *en passant*, that the *o.g.f.* can also be written in terms of the Ψ (or Digamma) function ($\Psi(z) := (\log\Gamma(z))'$)

$$G(n,z) \;=\; \frac{1}{z}\left(\Psi\left(-\frac{1}{z}\right) \;-\; \Psi\left(n+1-\frac{1}{z}\right) \;-\; 1\right)\;. \tag{6}$$

Because the *o.g.f.* for the column no. n of the *Stirling2* triangle **S2** A048993 without leading zeros is $\frac{1}{\prod_{j=1}^{n}(1-j\,x)} \;=\; \sum_{m=0}^{\infty} S2(m+n,n)\,x^m$ (see *e.g.*, [2], p. 298, Theorem 8.10), one has from eq. 4, after comparing coefficients of x^k,

$$\Sigma\,n^k \;\equiv\; n\,S_k(n) \;:=\; \sum_{m=0}^{min(n-1,k)} P(n,m)\,S2(k+n-m,n)\;, \tag{7}$$

with the coefficients $P(n,m) \;=$ A196837(n,m) of the row polynomial $P(n,x)$. (Remember that the (infinite) matrix **S2** has zeros above the main diagonal (it is lower triangular), therefore the upper limit of the sum is as given.) This **P** triangle organizes the sum over the kth power of the first n positive integers in terms of the nth column of the *Stirling* triangle **S2** read backwards starting with row no. $k+n$.

From eq. 5 it is clear that one also has

$$P(n,x) \;=\; \sum_{j=1}^{n}\prod_{\substack{l=1\\ l\neq j}}^{n}(1\;-\;l\,x)\;. \tag{8}$$

This shows explicitly that $P(n,x)$ is a polynomial having degree $n-1$. Now the elementary symmetric functions $\sigma_m(1,2,...,n)$ enter the stage because they are given here by $\prod_{j=1}^{n}(1\;-\;j\,x) \;=:\; \sum_{m=0}^{n}(-1)^m\,\sigma_m(1,2,...,n)\,x^m$ with $\sigma_0 \;=\; 1$. From this expansion $\sigma_m(1,2,...,n) \;=\; \sum_{1\leq a_1<a_2<...<a_m\leq n} a_1\,a_2\,\cdots a_m$, with $\binom{n}{m}$ terms. Now this is, in fact, $|S1(n+1,n+1-m)|$, as one can prove by mapping this problem to the combinatoric interpretation of the *Stirling* numbers of the first kind **S1** as cycle counting numbers of permutations. The signed lower triangular matrix **S1** is given in A048994. For a proof of this see [7], p.19, *Second proof.* In the jth term of the sum of eq. 8 the number j is excluded from the product. Therefore one gets elementary symmetric expressions for $n-1$ numbers. However, one does not have to go into these sums in detail, because by a symmetry and counting argument one is led immediately to the result for $P(n,m)$, the coefficient of x^m of $P(n,x)$. Each of the n product terms in the sum of eq. 8, when written in terms of the elementary symmetric function $\sigma_m(1,2,...,\not{j},...,n)$, for $m\in\{0,1,...,n-1\}$, has $\binom{n-1}{m}$ terms. Altogether (summed over j) there are $n\binom{n-1}{m}$ (signed) terms with products of m numbers multiplying x^m. Each product with m factors from all the numbers from $\{1,2,...,n\}$ appears, even though in the individual j-th term from eq. 8 one number, namely j, was missing. It is clear by symmetry that for each of these distinct products the multiplicity with which it appears has to be the same. Therefore one finds this multiplicity number M

from the equation $n\binom{n-1}{m} = M\binom{n}{m}$, the latter binomial being the number of terms of $\sigma_m(1,2,...,n)$. Therefore $M = n-m$, and this proves that the triangle $P(n,m) =$ A196837(n,m) is given by

$$P(n,m) = (-1)^m\,(n-m)\,|S1(n+1,n+1-m)| = (n-m)\,S1(n+1,n+1-m)\,. \tag{9}$$

This leads to the following formula for the sums of powers of positive integers.

$$\boxed{\Sigma\, n^k \equiv n\,S_k(n) = \sum_{m=0}^{min(k,n-1)} (n-m)\,S1(n+1,n+1-m)\,S2(n+k-m,n)\;, n\in\mathbb{N}, k\in\mathbb{N}_0\,.} \tag{10}$$

To the knowledge of the author this is a novel formula. In the *Figure* this product is illustrated, and the example $n = 5,\ k = 3$ is given. For $k = 1$ this is true due to the fact that $-S1(n+1,n) = \frac{(n+1)\,n}{2} = S2(n+1,n)$ which follows, *e.g.*, from the recurrence relations.

Two known formulae expressing $\Sigma\, n^k$ in terms of *Stirling*2 numbers and binomials are due to *Knuth* [4], p. 285, and they are for $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$

$$(i)\quad \Sigma\, n^k = \delta_{k,0}\,n + \sum_{m=1}^{k} m!\,S2(k,m)\binom{n+1}{m+1}\,, \tag{11}$$

$$(ii)\quad \Sigma\, n^k = \sum_{m=0}^{k} (-1)^{k-m}\,m!\,S2(k,m)\binom{n+m}{m+1}\,. \tag{12}$$

It is clear that the method of this note can be applied also to alternating sums of powers.

The author would appreciate information on the literature covering this *o.g.f.* $G(n,x)$, the used reverse *p.f.d.* and formula eq. 10.

For a short historical account on these sums of powers of integers see *Edwards* [3] and *Knuth* [4], where also further references are found. See also A093556. The references to books by *Ivo Schneider* and *Kurt Hawlitschek* on *Johannes Faulhaber* (1580-1635) are found there and in A093645.

Addendum, Oct 23 2011: Power sums as polynomials in n

In order to obtain $\Sigma\, n^k$ as a polynomial in n one can use, as done in the derivation of eq. (12), first a basis change from n^k to rising factorials $n^{\overline{l}}$ (see. e.g. the *Graham et al.* reference given under A196838, eq. (6.12), p. 249), then sum, using the fundamental identity $\sum_{l=1}^{n} \frac{l^{\overline{k}}}{k!} = \frac{n^{\overline{k+1}}}{(k+1)!}$ (a standard binomial formula). In this way one derives *Knuth*'s eq. (12). Now one transforms back from rising factorials to the power basis with the help of *Stirling1* numbers (see, e.g., the mentioned *Graham et al.* reference, eq. (6.13), p. 249), and finds the following formula.

$$\Sigma\, n^k = \sum_{m=1}^{k+1} r(k,m)\,n^m,\ k\in\mathbb{N}_0,\ n\in\mathbb{N},\ \text{with the rational number triangle}$$

$$r(k,m) := (-1)^{k+1-m}\sum_{l=m-1}^{k} S2(k,l)\,S1(l+1,m)\,\frac{1}{l+1}\,, \tag{13}$$

with the *Stirling* numbers of the first and second kind $S1$ and $S2$, found under A048994 and A048993, respectively. This rational triangle $r(n,m)$ is given under A196840(k,m)/A162299$(k+1,m)$. There one can also find the standard formula for $r(k,m)$ in terms of *Bernoulli* numbers $B(n) =$

A027641(n)/A027642(n). This leads to an identity expressing $\frac{B(k+1-m)}{k+1-m}\binom{k}{m}$, for $m = 1, ..., k-1$, by this sum over the *Stirling* numbers eq. (13).

Addendum, Oct 31 2011: Alternating power sums

As mentioned above, the *o.g.f.* of alternating power sums of positive integrs can be found similarly. For each $n \geq 1$ one defines the alternating power sums as

$$\hat{\Sigma}\, n^k \;=\; \sum_{j=1}^{n} (-1)^{n-j}\, j^k \;,\;\; n \,\in\, \mathbb{N},\;\; k \,\in\, \mathbb{N}_0 \;. \tag{14}$$

We mention, *en passant*, the well known result for $k \geq 1$, [1], p. 804, 23.1.4. (it is clear that $\hat{\Sigma}\, n^k$ vanishes for $k = 0$ if n is even , and it is 1 for odd n).

$$\hat{\Sigma}\, n^k \;=\; \frac{1}{2}\,\left(E_k(n+1) \,+\, (-1)^n\, E_k(0)\right)\,,\;\; n \,\in\, \mathbb{N},\;\; k \,\in\, \mathbb{N}\;, \tag{15}$$

with the *Euler* polynomials $E_k(x)$, whose rational coefficients are shown in A060096/A060097.

The *e.g.f.* is defined by $\hat{g}(n,x) \,:=\, \sum_{k=0}^{\infty} \hat{\Sigma}\, n^k\, \frac{x^k}{k!}$, and it is obvously given by

$$\hat{g}(n,x) \;=\; \sum_{j=1}^{n} (-1)^{n-j}\, e^{j\,x} \;=\; e^x\, \frac{(-1)^{n-1} \,+\, e^{n\,x}}{1 \,+\, e^x}\;. \tag{16}$$

The corresponding *o.g.f.* $\hat{G}(n,x) \,:=\, \sum_{k=0}^{\infty} \hat{\Sigma}\, n^k\, x^k$ is found *via Laplace* transformation, like in the main part of this note, and it is

$$\hat{G}(n,x) \;=\; \sum_{j=1}^{n} (-1)^{n-j}\, \frac{1}{1-j\,x} \;=\; \frac{\hat{P}(n,x)}{\prod_{j=1}^{n}(1-j\,x)}\,, \tag{17}$$

with the numerator polynomials $\hat{P}$ given by

$$\hat{P}(n,x) \;=\; (-1)^{n-1} \sum_{j=1}^{n} (-1)^{j+1} \prod_{\substack{l=1\\ l\neq j}}^{n} (1 \,-\, l\,x). \tag{18}$$

Now one treats the even and odd n case separately.

Even n case ($\mathbf{n \,=\, 2\,p}$):

It is clear that for even n the x^0 term vanishes because of the alternating sum. Maple [5] shows that one should also extract a factor p, and this leads to the following *Ansatz*.

$$\hat{G}(2\,p,x) \;=\; \frac{p\,x\,Qe(p,x)}{\prod_{j=1}^{2\,p}(1-j\,x)}\;, \tag{19}$$

with $Qe(p,x)$ given by

$$Qe(p,x) \;=\; \frac{-1}{p\,x} \sum_{j=1}^{2\,p} (-1)^{j+1} \prod_{\substack{l=1\\ l\neq j}}^{2\,p} (1 \,-\, l\,x)\;. \tag{20}$$

This is a polynomial of degree $2p - 1 - 1 = n - 2$. In order to obtain a formula for the coefficients $Qe(n,m) := [x^m]\, Qe(p,x)$ one considers the coefficient of x^{m+1}, for $m \in \{0,1,...,2(p-1)\}$ in the $j-$sum for $-p\,x\,Qe(p,x)$ (the term for $m = -1$ vanishes, as mentioned above). Group together pairs of consecutive terms in this $j-$sum, namely the terms for $j = 2\,i - 1$ and $j = 2\,i$, for $i = 1,2,..,p$. In each of these pairs the terms from the elementary symmetric functions σ_{m+1} neither with factor $2\,i - 1$ nor with $2\,i$ cancel, and thus the remaining positive terms of σ_{m+1} have a factor $2\,i$ but not $2\,i - 1$ (because of $l \neq j = 2\,i - 1$). Similarly the left over corresponding negative terms have no factor $2\,i$ but the factor $2\,i - 1$. Therefore one can combine these pairs to produce $(2\,i - (2\,i - 1))\,\sigma_m(1,2,..,\mathrm{no}(\,2\,i - 1, 2\,i),...,2\,p) = \sigma_m(1,2,..,\mathrm{no}(\,2\,i - 1, 2\,i),...,2\,p)$. Thus one is led to the elementary symmetric functions σ_m with two adjacent numbers omitted. The family of number triangles for such functions will be called $S_{i,j}(n,m)$, for $1 \leq i < l \leq n$, in the general case. Here $i \mapsto 2\,i - 1, j \mapsto 2\,i$. In order to have triangles one takes for $n < i$ the usual elementary symmetric functions $\sigma_m(n)$, and for $n \geq i$ one defines $S_{i,j}(n,m) := \sigma_m(1,2,...,\not{i},...,\not{j},...,n+2)$. This guarantees that each term has n factors. The triangles $S_{1,2}(n,m)$ and $S_{3,4}(n,m)$ are shown in [6] as A196845 and A196846, respectively (m is there called k). The entries of these triangles are expressed in terms of triangles of the type $S_j(n,m)$with the number j omitted in σ_m, which, in turn, are found from the *Stirling* numbers of the first kind. With the notation of these number triangles one finds $Qe(n,m)$, for $m \in \{0,1,...,2(p-1)\}$ (after multiplication with $\frac{-(-1)^m}{p}$, remembering that the x in the denominator has been acounted for by considering coefficients of x^{m+1}), with the following result.

$$Qe(n,m) \;=\; (-1)^m\,\frac{1}{p}\,\sum_{i=1}^{p} S_{2\,i-1,2\,i}(2(p-1),m)\ . \tag{21}$$

This number triangle is given in A196848, and the polynomials $Qe(p,x)$ are shown for $p = 1,...,10$ in *Table 2.*

Odd n case (n = 2 p + 1):

In the odd n case it is clear that the coefficient of x^0 in $Qo(p,x) := \hat{P}(2\,p+1,x)$, for $p \in \{0,1,...,2\,p = n-1\}$ is always 1. Extracting coefficients of $(-1)^m\,x^m$ in the $j-$sum in eq. (17) for $n = 2p+1$, one proceeds like above by considering pairs of consecutive odd and even js, with the last term, the one for $j = 2p+1$, left unpaired. This last term is the elementary symmetric function $\sigma_m(1,2,...,2\,p) = |S1(2\,p+1, 2\,p+1-m)|$. With the definition of the number triangles $S_{2\,i\,-\,1,2\,i}(n,m)$ given above the result for $Qo(p,m) = [x^m]\,\hat{P}(2\,p+1,x)$ becomes

$$\begin{cases} 1 & \text{if } \ m = 0\ , \\ (-1)^m \left(\displaystyle\sum_{i=1}^{p} S_{2\,i-1,2\,i}(2\,p+1, m-1) + |S1(2\,p+1, 2\,p+1-m)| \right) & \text{if } \ m \in \{1,2,...,2\,p\}\ . \end{cases} \tag{22}$$

This number triangle is given in A196847, and the polynomials $Qo(p,x) := \hat{P}(2\,p+1,x)$ are shown for $p = 0,...,9$ in *Table 3.*

Addendum, Nov 01 2011: O.g.f.s for fixed powers and Eulerian numbers

The *o.g.f.* $\tilde{G}(k,x) := \sum_{n=1}^{\infty} \Sigma\,n^k\,x^n$ can be computed using the so called *Worpitzky* identity involving the *Euler*ian numbers $E(n,m)$ shown in A173018. For this identity and the hint to use it for power sums see the Graham et al. reference given under A196838, eq. (6.37) on p. 255. The formula for the power sums is

$$\Sigma\,n^k \;=\; \sum_{p=0}^{k} E(k,p) \binom{n+p+1}{k+1} \;-\; \delta_{k,0}\ , \tag{23}$$

with the *Euler*ian number triangle $E(n,m)$ and the *Kronecker* δ symbol. From this one finds the o.g.f. in terms of the row polynomials, the *Euler*ian polynomials.

$$\tilde{G}(k,x) \;=\; \frac{x}{(1-x)^{k+2}}\, Eulerian(k,x)\ ,\ k\,\in\,\mathbb{N}_0\ . \tag{24}$$

This formula has been given by *Vladeta Jovovic* in a comment in the formula section of A000538. He also gave the *e.g.f.* for these *o.g.f.* s.

The author would like to thank Gary Detlefs for comments, and for pointing out some typos.

References

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[4] D. E. Knuth, "Johann Faulhaber and Sums of Powers", *Maths. of Computation*, 61 203 (1993) 277-294, also available under http://arxiv.org/abs/math/9207222

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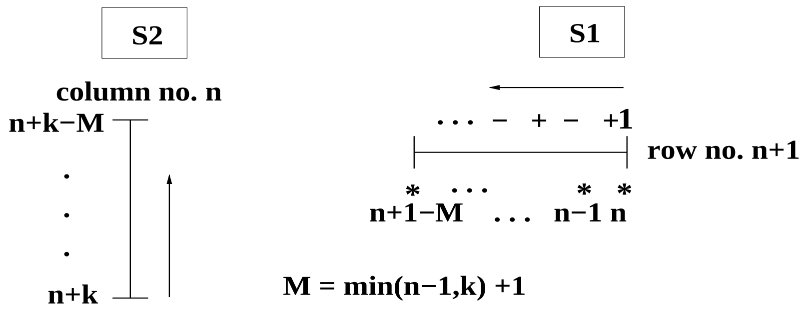
AMS MSC numbers: 05A15, 11B73, 11C08

OEIS A-numbers:A000538, A027641, A027642, A048993, A048994, A053154, A060096, A060097, A093556, A093645, A103438, A162299, A173018, A196837, A196838, A198640, A196845, A196845, A198628.

Figure

Sums of powers of the first n positive integers



Example: n=5, k=3, M= min(4,3)+1 = 4

1*5*1050 − 15*4*140 + 85*3*15 − 225*2*1 = 225

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 1 + 8 + 27 + 64 + 125 = 225$

Table 1 : Row polynomials of A196837 for $n = 1, 2, ..., 15$.

n	$P(n,x)$
1	1
2	$2-3x$
3	$3-12x+11x^2$
4	$4-30x+70x^2-50x^3 = 2(2-5x)(1-5x+5x^2)$
5	$5-60x+255x^2-450x^3+274x^4$
6	$6-105x+700x^2-2205x^3+3248x^4-1764x^5 =$ $(2-7x)(3-42x+203x^2-392x^3+252x^4)$
7	$7-168x+1610x^2-7840x^3+20307x^4-26264x^5+13068x^6$
8	$8-252x+3276x^2-22680x^3+89796x^4-201852x^5+236248x^6-109584x^7 =$ $4(2-9x)(1-27x+288x^2-1539x^3+4299x^4-5886x^5+3044x^6)$
9	$9-360x+6090x^2-56700x^3+316365x^4-1077300x^5+2171040x^6-2345400x^7+1026576x^8$
10	$10-495x+10560x^2-127050x^3+946638x^4-4510275x^5+13667720x^6-25228500x^7+$ $25507152x^8-10628640x^9 =$ $(2-11x)(5-220x+4070x^2-41140x^3+247049x^4-896368x^5+1903836x^6-2143152x^7+$ $966240x^8)$
11	$11-660x+17325x^2-261360x^3+2501961x^4-15825348x^5+66697675x^6-183982920x^7+$ $315774228x^8-301835952x^9+120543840x^{10}$
12	$12-858x+27170x^2-501930x^3+5995704x^4-48486438x^5+269941386x^6-1030350750x^7+$ $2628827344x^8-4242044664x^9+3863119104x^{10}-1486442880x^{11} =$ $2(2-13x)(3-195x+5525x^2-89570x^3+916721x^4-6162923x^5+27426347x^6-79316432x^7+$ $141650028x^8-139785984x^9+57170880x^{10})$
13	$13-1092x+41041x^2-910910x^3+13270257x^4-133357224x^5+945255311x^6-4745658918x^7+$ $16680593930x^8-39830815024x^9+60941259288x^{10}-53193434112x^{11}+19802759040x^{12}$
14	$14-1365x+60060x^2-1576575x^3+27497470x^4-335810475x^5+2947292920x^6-$ $18770176425x^7+86455937568x^8-283316833800x^9+638886422720x^{10}-932967781200x^{11}+$ $784313595648x^{12}-283465647360x^{13} =$ $(2-15x)(7-630x+25305x^2-598500x^3+9259985x^4-98455350x^5+735231335x^6-$ $3870853200x^7+14196569784x^8-35184143520x^9+55562134960x^{10}-49767878400x^{11}-$ $18897709824x^{12})$
15	$15-1680x+85540x^2-2620800x^3+53895842x^4-785584800x^5+8352861660x^6-$ $65661024000x^7+382417906871x^8-1636819264080x^9+5048360535400x^{10}-$ $10827253382400x^{11}+15170987111472x^{12}-12331635229440x^{13}+4339163001600x^{14}$
⋮	

Table 2 : Row polynomials of A196848 p = 1,...,10

$\mathbf{p = n/2}$	$\mathbf{Qe(p, x)}$
1	1
2	$1 - 5\,x + 7\,x^2$
3	$1 - 14\,x + 73\,x^2 - 168\,x^3 + 148\,x^4$
4	$1 - 27\,x + 298\,x^2 - 1719\,x^3 + 5473\,x^4 - 9162\,x^5 + 6396\,x^6$
5	$1 - 44\,x + 830\,x^2 - 8756\,x^3 + 56453\,x^4 - 227744\,x^5 + 562060\,x^6 - 778800\,x^7 + 468576\,x^8$
6	$1 - 65\,x + 1865\,x^2 - 31070\,x^3 + 332463\,x^4 - 2385305\,x^5 + 11612795\,x^6 - 37875240\,x^7 + 79269676\,x^8 - 96420480\,x^9 + 52148160\,x^{10}$
7	$1 - 90\,x + 3647\,x^2 - 87900\,x^3 + 1402023\,x^4 - 15575130\,x^5 + 123448001\,x^6 - 702763920\,x^7 + 2849969416\,x^8 - 8027712480\,x^9 + 14918150352\,x^{10} - 16460801280\,x^{11} + 8203541760\,x^{12}$
8	$1 - 119\,x + 6468\,x^2 - 212653\,x^3 + 4720646\,x^4 - 74781147\,x^5 + 870968684\,x^6 - 7569404479\,x^7 + 49281440145\,x^8 - 238993012874\,x^9 + 849263860648\,x^{10} - 2143751307768\,x^{11} + 3635508507408\,x^{12} - 3714722544960\,x^{13} + 1733641056000\,x^{14}$
9	$1 - 152\,x + 10668\,x^2 - 458584\,x^3 + 13503966\,x^4 - 288617448\,x^5 + 4627515940\,x^6 - 56727541672\,x^7 + 536863254585\,x^8 - 3931950087968\,x^9 + 22191960382536\,x^{10} - 95428928224224\,x^{11} + 306299819370448\,x^{12} - 709182345858432\,x^{13} + 1117412056889856\,x^{14} - 1072199396459520\,x^{15} + 473875121664000\,x^{16}$
10	$1 - 189\,x + 16635\,x^2 - 905436\,x^3 + 34130706\,x^4 - 945559566\,x^5 + 19938286870\,x^6 - 326943834588\,x^7 + 4223081431941\,x^8 - 43254549907821\,x^9 + 351833219573295\,x^{10} - 2265342943068576\,x^{11} + 11450188172985976\,x^{12} - 44781233983066224\,x^{13} + 132447580643617200\,x^{14} - 285758630338003200\,x^{15} + 423616834840939776\,x^{16} - 385562909165414400\,x^{17} + 162705528979660800\,x^{18}$
$\vdots$	

Example: The *o.g.f.* for the sequence $\{-(1^k - 2^k + 3^k - 4^k)\}_{k=0}^{\infty}$ $(p = 2, n = 4)$, found in 2*A053154, is

$$Ge(2, x) \;=\; \frac{2\,x\,(1 - 5\,x + 7\,x^2)}{\prod_{j=1}^{4}(1 - j\,x)}.$$

Table 3 : Row polynomials of A196847 p = 0,...,9

$p = \frac{n-1}{2}$	$Qo(p, x)$
0	1
1	$1 - 4\,x + 5\,x^2$
2	$1 - 12\,x + 55\,x^2 - 114\,x^3 + 94\,x^4$
3	$1 - 24\,x + 238\,x^2 - 1248\,x^3 + 3661\,x^4 - 5736\,x^5 + 3828\,x^6$
4	$1 - 40\,x + 690\,x^2 - 6700\,x^3 + 40053\,x^4 - 151060\,x^5 + 351800\,x^6 - 465000\,x^7 + 270576\,x^8$
5	$1 - 60\,x + 1595\,x^2 - 24720\,x^3 + 247203\,x^4 - 1665900\,x^5 + 7660565\,x^6 - 23745720\,x^7 + 47560876\,x^8 - 55805520\,x^9 + 29400480\,x^{10}$
6	$1 - 84\,x + 3185\,x^2 - 72030\,x^3 + 1081353\,x^4 - 11344872\,x^5 + 85234175\,x^6 - 461800710\,x^7 + 1790256286\,x^8 - 4843901664\,x^9 + 8693117160\,x^{10} - 9320129280\,x^{11} + 4546558080\,x^{12}$
7	$1 - 112\,x + 5740\,x^2 - 178304\,x^3 + 3747982\,x^4 - 56355936\,x^5 + 624649940\,x^6 - 5180978432\,x^7 + 32290710473\,x^8 - 150403364272\,x^9 + 515162381720\,x^{10} - 1258326123264\,x^{11} + 2073788193744\,x^{12} - 2069274574080\,x^{13} + 948550176000\,x^{14}$
8	$1 - 144\,x + 9588\,x^2 - 391608\,x^3 + 10974894\,x^4 - 223638408\,x^5 + 3425288452\,x^6 - 40195145304\,x^7 + 364960154409\,x^8 - 2570591813832\,x^9 + 13988743440672\,x^{10} - 58158727694928\,x^{11} + 181015904743696\,x^{12} - 407711994791616\,x^{13} + 627139182204288\,x^{14} - 589805676956160\,x^{15} + 256697973504000\,x^{16}$
9	$1 - 180\,x + 15105\,x^2 - 784800\,x^3 + 28275306\,x^4 - 749742840\,x^5 + 15153672490\,x^6 - 238561930800\,x^7 + 2963426487261\,x^8 - 29242932326100\,x^9 + 229608908058405\,x^{10} - 1430012790032400\,x^{11} + 7006810619981656\,x^{12} - 26626572692739360\,x^{13} + 76710622505994000\,x^{14} - 161648143661520000\,x^{15} + 234739505890123776\,x^{16} - 209987960948075520\,x^{17} + 87435019510272000\,x^{18}$
⋮	

Example: The *o.g.f.* for the sequence $\{1^k - 2^k + 3^k - 4^k + 5^k\}_{k=0}^{\infty}$ ($p = 2, n = 5$), found in A198628, is

$$Go(2, x) = \frac{(1 - 12\,x + 55\,x^2 - 114\,x^3 + 94\,x^4)}{\prod_{j=1}^{5}(1 - j\,x)}.$$