Maple-assisted proof of empirical generating function for A188270

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For $n \geq 1$, $a(n)$ is the coefficient of y^0 in the Laurent series of the coefficient of x^n in

$$G(x, y) = \prod_{j=-2}^{2} \left(\frac{1}{1 - x\, y^{j^3}} \right)$$

> $G := \prod_{j=-2}^{2} \left(\frac{1}{1 - x\, y^{j^3}} \right)$

$$G := \frac{1}{\left(1 - \frac{x}{y^8}\right) \left(1 - \frac{x}{y}\right) (1 - x) (-x y + 1) (-x y^8 + 1)} \qquad \textbf{(1)}$$

It will be convenient for this analysis to include the term $a(0) = 1$. Here is the partial fraction expansion of $G(x, y)$ with respect to x:

> $convert(G, parfrac, x)$

$$-\frac{y^9}{(y^8 - 1)^2 (y - 1)^2 (-1 + x)} - \frac{y^8}{(y^7 - 1) (y^8 - 1) (y^9 - 1) (y^{16} - 1) (-y^8 + x)} + \frac{y^8}{(y^7 - 1) (y - 1) (y^2 - 1) (y^9 - 1) (-y + x)} + \frac{y^{12}}{(y^9 - 1) (y^2 - 1) (y - 1) (y^7 - 1) (x y - 1)} - \frac{y^{40}}{(y^{16} - 1) (y^9 - 1) (y^8 - 1) (y^7 - 1) (x y^8 - 1)} \qquad \textbf{(2)}$$

The terms with $-1 + x$, $x y - 1$ or $x y^8 - 1$ in the denominator, when expanded in powers of x and then powers of y, will contain only positive powers of y. The terms we want are these:

> $G_1 := op(2, \textbf{(2)})$

$$G_1 := -\frac{y^8}{(y^7 - 1) (y^8 - 1) (y^9 - 1) (y^{16} - 1) (-y^8 + x)} \qquad \textbf{(3)}$$

> $G_2 := op(3, \textbf{(2)})$

$$G_2 := \frac{y^8}{(y^7 - 1) (y - 1) (y^2 - 1) (y^9 - 1) (-y + x)} \qquad \textbf{(4)}$$

The series for G_1 and G_2 in powers of x are

> $T_1 := convert(G_1, FPS, x)$

$$T_1 := \sum_{k=0}^{\infty} \frac{\left(\frac{1}{y^8}\right)^k x^k}{(y^7-1)(y^8-1)(y^9-1)(y^{16}-1)} \tag{5}$$

> $T_2 := convert(G_2, FPS, x)$

$$T_2 := \sum_{k=0}^{\infty} \left(- \frac{y^7 \left(\frac{1}{y}\right)^k x^k}{(y^7-1)(y-1)(y^2-1)(y^9-1)} \right) \tag{6}$$

Thus $a(n) = b(8\,n) + c(n)$ where $b(n)$ and $c(n)$ have ordinary generating functions $B(y)$ and $C(y)$ as follows:

> $B := eval(op(1, T_1), k=0)$

$$B := \frac{1}{(y^7-1)(y^8-1)(y^9-1)(y^{16}-1)} \tag{7}$$

> $C := eval(op(1, T_2), k=0)$

$$C := - \frac{y^7}{(y^7-1)(y-1)(y^2-1)(y^9-1)} \tag{8}$$

Note that if $g(z) = \sum_{n=0}^{\infty} b(n)\,y^n$, $\sum_{n=0}^{\infty} b(8\,n)\,y^{8\,n} = \frac{1}{8} \sum_{j=0}^{7} g(\omega^j y)$ where ω is a primitive 8'th root of unity, i.e. a 4'th root of -1. So here is the generating function of $b(8\,n)$:

> $B8 := eval\left(simplify\left(\frac{1}{8} \cdot add(eval(B, y=\omega^j y), j=0..7), \{\omega^4=-1\}\right), y=y^{\frac{1}{8}}\right)$

$$B8 := \frac{(y^2+1)(y^4+1)(y^8+1)}{y^{19}-y^{18}-y^{17}+y^{16}-y^{12}+y^{11}+y^8-y^7+y^3-y^2-y+1} \tag{9}$$

Adding this to $C(y)$ gives us the generating function of $a(n)$:

> $normal(C + B8)$

$$\frac{y^8-y^7+y^6-y^5+y^4-y^3+y^2-y+1}{y^{13}-2\,y^{12}+2\,y^{10}-y^9-y^4+2\,y^3-2\,y+1} \tag{10}$$

Here was the "empirical" generating function, which was for the sequence with offset 1 :

> $empirical := x*(1+x-3*x\text{^}2+2*x\text{^}3-x\text{^}4+x\text{^}5-x\text{^}6+x\text{^}7+x\text{^}8-2*x\text{^}9+2*x\text{^}11-x\text{^}12)/((1-x)\text{^}4*(1+x)*(1+x+x\text{^}2)*(1+x\text{^}3+x\text{^}6))$

$$empirical := \frac{x\,(-x^{12}+2\,x^{11}-2\,x^9+x^8+x^7-x^6+x^5-x^4+2\,x^3-3\,x^2+x+1)}{(1-x)^4\,(1+x)\,(x^2+x+1)\,(x^6+x^3+1)} \tag{11}$$

The difference between these is our term $a(0) = 1$.

> $normal(C + B8 - 1 - subs(x=y, empirical))$

$$0 \tag{12}$$

This completes the proof.