A187988: EXPONENTIAL DIOPHANTINE EQUATION WITH SUMS OF POWERS OF 2

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Abstract. We discuss aspects of OEIS sequence A187988, enumerating balanced sums of powers of two, explain the mechanics of the Maple implementation, and provide conjectural rational generating functions along rows and columns of the table.

1. Statement of the Combinatorial Problem

1.1. **Exponential diophantine equation.** The sequence is concerned with counting solutions to Hardin's equation [2, A187988]

$$\sum_{i=1}^{n} \operatorname{sgn}(x_i) 2^{|x_i|} = 0, \tag{1}$$

where the sign function is defined as

$$\operatorname{sgn}(m) \equiv \begin{cases} 1, & m \geq 0; \\ -1, & m < 0. \end{cases} \tag{2}$$

To obtain a finite number of solutions, the exponents are limited to be in a predefined range $-E \leq x_i \leq E$. The theme is to find all multisets of exponents x_i^+, $0 \leq x_i^+ \leq E$, and a multiset of exponents x_i^-, $0 < x_i^- \leq E$ for the exponential diophantine equation

$$2^{x_0^+} + 2^{x_1^+} + 2^{x_2^+} + \cdots + 2^{x_{n^+}^+} = 2^{x_1^-} + 2^{x_2^-} + \cdots + 2^{x_{n^-}^-}, \tag{3}$$

where the total number of terms on both sides is some predefined $n^+ + n^- = n$. *Multiset* means the exponents $x_i^\pm$ do not need to be distinct.

1.2. **Linear diophantine equation.** Any solution can be encoded as a vector of frequencies f_i^+ and f_i^- counting how often exponents appear on the left and right sides of this equation:

$$\sum_{i\geq 0} f_i^+ 2^i = \sum_{i \geq 1} f_i^- 2^i. \tag{4}$$

The asymmetry in the start index in the sums of both sides is a result of putting the zero in the bag of numbers with positive sign in (2). For convenience of the nomenclature, we place the sum of the nonnegative x_i on the left hand side (LHS) of the equation, the sum of the negative x_i on the right hand side (RHS).

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A standard notation of partitions denotes how often (as an exponent) a number (at the base) occurs in a partition; equation (4) could be written as a formal equality between two weak partitions of n:

(5)
$$0^{f_0^+}1^{f_1^+}2^{f_2^+}3^{f_3^+}\dots E^{f_{n^+}^+} \sim 1^{f_1^-}2^{f_2^-}3^{f_3^-}\dots E^{f_{n^-}^-};\quad \sum f_i^+ + \sum f_i^- = n,\quad 0\le f_i^+, f_i^-.$$

Example 1. *Given $E=6$ and $n=8$, the solution $x_i=\{-6,-2,0,0,1,4,4,5\}$ satisfying $2^0+2^0+2^1+2^4+2^4+2^5=2^6+2^2$ has the partition representation $0^2 1^1 4^2 5^1 \sim 2^1 6^1$.*

Definition 1. *$T(n,E)$ is the number of solutions to the diophantine equation (4) under the constraints*

$$\sum_{i=0}^{E} f_i^+ + \sum_{i=1}^{E} f_i^- = n;\quad 0\le f_i^+, f_i^-. \tag{6}$$

Definition 2. *A set of solutions $\{x_i\}$ satisfying (1) or an equivalent set of frequencies $f_i^\pm$ satisfying (5) is called balanced.*

For any fixed limit E, (4) is a linear diophantine equation [5, 3, 1, 6].

1.3. **Abacus.** A physical realization of these sets $f_i^\pm$ is a 2-sided abacus with pegs enumerated $0,1,\dots,E$ and stacks of $f_0^+, f_1^+,\dots,f_E^+$ tokens on the LHS, and pegs enumerated $1,2,\dots,E$ and stacks of $f_1^-, f_2^-,\dots,f_E^-$ tokens on the RHS. [Think of this as two towers of Hanoi with indistinguishable tokens...] There are moves on this abacus which keep the stacks balanced (in the arithmetic sense defined above), for example:

- Removing two tokens on peg i and adding one token on peg $i+1$ on the same side (which decreases n by 1) or in reverse (which increases n by 1). The arithmetic equivalent is $2\times 2^i = 2^{i+1}$. The standard binary representation of a number is actually found by repeating that move as often as possible in any order, which means as long as there are pegs with 2 or more tokens; the final position is a representation were all $f_i \le 1$ on that side, f_i representing the bits of the binary representation.
- Adding a token on peg i of both sides (which increases n by 2) or removing one on peg i of both sides (which decreases n by 2). The arithmetic equivalent is adding 2^i to LHS and RHS.

2. Primitive Solutions

The primitive solutions are those where the multisets of exponents on both sides of Equation (3) are the same, $f_i^+ = f_i^-$, for all $i\ge 1$. This implies that the exponent 0 does not appear on the left hand side: $f_0^+=0$. They can be counted given the mere constraints on the total number terms, n, and the maximum, E, of the exponents. There are $n/2$ terms on each side of the equation, so n is even. Considering the ordered list of exponents and their frequencies in the notation (5), the frequencies are weak compositions of $n/2$ into E terms. So the number of primitive solutions is [8, §1.2]

$$T^p(n,E) = \begin{cases} 0, & n\,\text{odd};\\ \binom{n/2+E-1}{n/2}, & n\,\text{even}.\end{cases} \tag{7}$$

This provides a trivial lower bound $T(n,E)\ge T^p(n,E)$ on the number of solutions.

$i\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17							
0	0	1																							
1	0	1	1																						
2	0	1	1	1	1																				
3	0	1	1	1	2	2	1	1	1																
4	0	1	1	1	2	3	4	4	4	4	3	2	2	2	1	1	1								
5	0	1	1	1	2	3	5	8	10	11	13	13	12	13	13	12	12	11	9	8	7	6	6	5	4

Table 1. Number of ways of partitioning 2^i into $0 \le k \le 2^i$ powers of 2 [2, A089052].

3. Small Parameters

For small total number n of terms on both sides, the number of solutions can be counted by hand.

3.1. n=0.

$n = 0$ is not interesting because there are no solutions besides admitting $0 = 0$, $n^+ = n^- = 0$, $T(0, E) = 1$.

3.2. n=1.

For $n = 1$ there is one term on one side, none on the other, and because powers of 2 are positive, there are no solutions,

$$T(1, E) = 0. \tag{8}$$

3.3. n=2.

Generally, there is at least one term on each side, because powers of 2 are positive, so balancing the sums requires some positive contribution on both sides. For $n = 2$ there must be one term on each side, $n^+ = n^- = 1$, so these are primitive solutions counted in (7):

$$T(2, E) = E. \tag{9}$$

3.4. n=3.

If $n = 3$,

- and $n^+ = 2$, $n^- = 1$, the sole term on the RHS is a power of 2 which is decomposed into two, not necessarily distinct powers of 2 on the LHS. Considering the base-2 representation of all 3 numbers involved, this can only be done in one way, where each term on the LHS is half the term on the RHS, see Table 1. The exponent on the RHS is from 1 to E, so there are E solutions of this form.
- or $n^+ = 1$, $n^- = 2$. This is basically the same set of solutions considered in the previous item (by swapping the terms on both sides), but the term 2^0 is not admitted to the LHS, so there are only $E - 1$ solutions of this form.

In total

$$T(3, E) = 2E - 1. \tag{10}$$

3.5. n=4.

If $n = 4$, it helps to place one or two tokens of the abacus on the same position $i \le E$ on both sides, and to generate more solutions by the left moves:

- and $n^+ = 3$, $n^- = 1$, the sole term on the RHS is a power of 2 which is decomposed into three, not necessarily distinct powers of 2 on the LHS. Considering the base-2 representation of all 3 numbers involved, there is one decomposition with the partition $(j-2)^2(j-1)^1 \sim j^1$ in the notation (5). This requires $2 \le j \le E$, which comprises $E - 1$ solutions if $E \ge 2$.

- or $n^+=2$, $n^-=2$. Considering the base-2 representations of all 4 numbers involved, these must be primitive solutions, of which there are $T^p(4,E)=E(E+1)/2$. This is the only contribution if $E=1$.
- or $n^+=1$, $n^-=3$. This is basically the same set of solutions considered in the first bullet by swapping both sides, but the case $j=0$ cannot be split, so this yields $E-2$ solutions.

The total of the three subcases is [2, A134227]

$$T(4,E)=(E-1)+E(E+1)/2+(E-2)=(E-1)(E+6)/2, E\ge 2 \tag{11}$$

with generating function

$$\sum_{E\ge 0} T(4,E)x^E = \frac{x(1+x-x^3)}{(1-x)^3} = 3+x-\frac{3-9x+5x^2}{(1-x)^3}. \tag{12}$$

$$T(4,E) = -3\binom{E}{0}+3\binom{E}{1}+\binom{E}{2}, \quad E\ge 2. \tag{13}$$

3.6. **E=1.** If $E=1$, there are f_1^- terms equal 2 on the RHS, which leaves f_0^+ and f_1^+ terms equal to 1 and 2 on the LHS, which demands $f_0^+ + f_1^+\times 2 = f_1^-\times 2$. [The cases $f_0^+=0$ are the symmetric solutions (7), none for odd n, one for even n.] Therefore $f_0^+ = 2(f_1^- - f_1^+)$ and with (6) $n-f_1^- - f_1^+ = 2(f_1^- - f_1^+)$, $\rightsquigarrow n = 3(f_1^- - f_1^+)+2f_1^+$. So $T(n,E)$ is the number of partitions of n into $f_1^- - f_1^+$ parts equal 3 and f_1^+ parts equal 2 [2, A103221],

$$T(n,1)=T(n-2,1)+\begin{cases}1, & 3\mid n\\ 0, & 3\nmid n\end{cases}, \quad T(1,1)=0, \quad T(2,1)=1, \tag{14}$$

The initial terms appeared already above. The generating function is

$$\sum_{n\ge 0} T(n,1)z^n = \frac{1}{(1-z^2)(1-z^3)}. \tag{15}$$

4. Recursive Enumeration

The main task is to determine the number of solutions $T(n,E)$ as illustrated in Table 2.

Remark 1. *Hardin's table* [2, A187988] *actually shifts the rows to the left discarding the first $n-2$ entries of row n; our columns are the antidiagonals of his table.*

The task is to solve (4),

$$f_0^+ + 2f_1^+ + 4f_2^+\cdots + 2^E f_E^+ = 2f_1^- + 4f_2^- + \cdots 2^E f_E^-. \tag{16}$$

Remark 2. *This demonstrates that all f_0^+ of the solutions are even.*

Eliminating f_0^+ with (6) helps to confine this to the subspace of constant n:

$$n-f_1^+-f_2^+-\cdots-f_E^+-f_1^--f_2^--\cdots-f_E^-+2f_1^++4f_2^+\cdots+2^Ef_E^+ = 2f_1^-+4f_2^-+\cdots 2^Ef_E^-. \tag{17}$$

$$n+(2^1-1)f_1^++(2^2-1)f_2^+\cdots+(2^E-1)f_E^+ = (2^1+1)f_1^-+(2^2+1)f_2^-+\cdots+(2^E+1)f_E^-. \tag{18}$$

Finding solutions recursively for fixed maximum E may be done by nibbling off the rightmost terms of both sides, $(2^E-1)f_E^+$ and $(2^E+1)f_E^-$. The outer double loop

$n \backslash E$	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	2	3	4	5	6	7	8	9	10	11	12
3	1	3	5	7	9	11	13	15	17	19	21	23
4	1	4	9	15	22	30	39	49	60	72	85	99
5	1	7	19	36	57	82	111	144	181	222	267	316
6	2	10	31	67	117	181	260	355	467	597	746	915
7	1	12	48	123	243	411	632	912	1257	1673	2166	2742
8	2	18	79	225	488	890	1452	2199	3158	4357	5825	7592
9	2	23	116	374	903	1797	3144	5040	7593	10920	15146	20404
10	2	28	166	609	1633	3543	6649	11285	17829	26706	38385	53379
11	2	36	240	972	2857	6720	13507	24280	40262	62870	93724	134652
12	3	45	330	1484	4794	12252	26443	50481	88078	143661	222458	330555
13	2	52	443	2220	7858	21755	50338	102023	187351	319332	513829	789895

TABLE 2. $T(n, E)$, the number of solutions.

is over all possible $0 \le f_E^+ \le n$ and $0 \le f_E^- \le n$ but noticing as a speedup that the sum is also limited by $f_E^+ + f_E^- \le n$.

Remark 3. *So the sum is over a triangular region over the two rightmost coefficients. During the recurrence that means the multi-sum is over the simplices in the space of the* $2E$ *coordinates. There is some similarity with Erhardt sums.*

Once the two $f_E^\pm$ are fixed, they are "combined" on the left side of the equation defining a new $n \to n + (2^E - 1)f_E^+ - (2^E + 1)f_E^-$, reduced maximum exponent $E \to E - 1$, and reduced number of available abacus tokens $\sum f_i^+ + \sum f_i^- \to \sum f_i^+ + \sum f_i^- - f_E^+ - f_E^-$ for the next stage at the recurrence. (This updated n may become negative for intermediate stages of the recurrence.) At the bottom of this recurrence reaching $E = 0$, a solution has been established if the final equation is $0 = 0$, whereas any nonzero value for n indicates no solution is found. A massive speedup is obtained by realizing that for any (signed) n the equation can only be kept balanced if there is a sufficient number of tokens and exponents left in the (yet undecided) terms $\sum_{i=1}^{E}(2^i - 1)f_i^+$ and $\sum_{i=1}^{E}(2^i + 1)f_i^-$. The maximum residual sum in these terms is obtained by piling all remaining $f_i^- + f_i^+$ tokens into the term $2^E + 1$ with the highest coefficient, so if $|n| > (2^E + 1)(\sum f_i^+ + \sum f_i^-)$ there are no solutions in the associated branch of the recurrence, and that branch of the tree of recurrences can be pruned.

Remark 4. *This algorithm is not just enumerative but also constructive, finding all solutions explicitly.*

5. The Vector Space of the Linear Diophantine Equation

Eliminating f_E^+ from (16) yields

$$2^E n = \sum_{i=0}^{E-1}(2^E - 2^i)f_i^+ + \sum_{i=1}^{E}(2^E + 2^i)f_i^- \tag{19}$$

which is an inhomogeneous linear diophantine equation with inhomogeneity $2^E n$, fixed coefficients $2^E \pm 2^i$ and indeterminates $f_i^\pm$. Rosser's algorithm [7] generates for fixed E a matrix W which contains

- in the top row a solution (sequence of $f_i^\pm$) which changes the inhomgeneity by one—not helpful because it involves negative $f_i^\pm$ and initial states $f_1^+ = f_1^- = 1$ for even $n = 2$ and $f_0^+ = 2, f_1^- = 1$ for odd $n = 3$ seem to be more suitable;
- and in the other rows sets of incremental $f_i^\pm$ which keep the equation balanced without changing n, i.e., solutions for the homogeneous equation.

Example 2. *For $E = 2$ the matrix is*

f_0^+	f_1^+	f_1^-	f_2^-
0	0	-1	1
0	1	1	-1
-2	0	1	0

Example 3. *For $E = 5$ the matrix is*

f_0^+	f_1^+	f_2^+	f_3^+	f_4^+	f_1^-	f_2^-	f_3^-	f_4^-	f_5^-
1	0	0	0	0	-1	-1	1	0	0
0	1	-1	0	0	1	-1	0	0	0
0	0	1	-1	0	0	1	-1	0	0
0	0	0	1	-1	0	0	1	-1	0
0	0	0	0	1	0	0	0	1	-1
-2	-1	1	0	0	2	1	-1	0	0
2	2	-2	0	0	-3	1	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	-2	2	0	0	0	-1	1	0
0	0	0	-2	2	0	0	0	-1	1

Example 4. *For $E = 6$ the matrix is*

f_0^+	f_1^+	f_2^+	f_3^+	f_4^+	f_5^+	f_1^-	f_2^-	f_3^-	f_4^-	f_5^-	f_6^-
1	0	0	0	0	0	-1	-1	1	0	0	0
0	1	-1	0	0	0	1	-1	0	0	0	0
0	0	1	-1	0	0	0	1	-1	0	0	0
0	0	0	1	-1	0	0	0	1	-1	0	0
0	0	0	0	1	-1	0	0	0	1	-1	0
0	0	0	0	0	1	0	0	0	0	1	-1
-2	-1	1	0	0	0	2	1	-1	0	0	0
2	2	-2	0	0	0	-3	1	0	0	0	0
0	0	-2	2	0	0	0	0	-1	1	0	0
0	0	0	-2	2	0	0	0	0	-1	1	0
0	0	0	0	-2	2	0	0	0	0	-1	1

All lines from the 2nd on correspond to generators an infinite group of operations on the 2-sided abacus that keep n constant. The caveat is that the allowed moves must keep all $f_i^\pm$ nonnegative all the time.

This is a vector space: any linear combination of lines 2 and further down is also a move of tokens that keeps n constant. f_E^+ is implied by (6), the negative sum of the other entries in a row.

6. Conjectures

Based on the numerical counts obtained from the recursive algorithm, a set of heuristic rational generating functions ensues. As (18) is partitioning n in some sense in parts of 2^i+1 and 2^i-1 (two types of parts 3 because the coefficient in front of f_2^+ and f_1^- is the same), and since $1/[(1-x^i)^\alpha(1-x^j)^\beta\cdots]$ is the generating function for partitions into α sizes of i, β sizes of $j\ldots$ [4], the format of the denominators of some generating functions of the columns is expected.

The associated C-finite recurrences are implicit by expanding the polynomials of z or x in the denominators [9].

Conjecture 1. *Column $E=2$:*

$$\sum_{n\ge 0} T(n,2)z^n = \frac{1+z^2+z^3+2z^4+z^5+z^6}{(1-z^2)(1-z^5)(1-z^3)^2} \tag{20}$$

Conjecture 2. *Column $E=3$:*

$$\sum_{n\ge 0} T(n,3)z^n = \frac{1-z+2z^2+z^3+3z^4+3z^5+4z^6+2z^7+4z^8+3z^9+3z^{10}+2z^{11}+3z^{12}+z^{14}}{(1-z^2)(1-z^5)(1-z^9)(1-z^3)^2(1-z)} \tag{21}$$

The generating functions for the rows all appear to have denominators $(1-x)^k$, so $T(n,E)$ appear to be polynomials in E for sufficiently large E.

Conjecture 3. *Row $n=5$:*

$$\sum_{E\ge 0} T(5,E)x^E = 8+4x+x^2-\frac{8-21x+9x^2}{(1-x)^3}. \tag{22}$$

$$T(5,E) = -8\binom{E}{0}+5\binom{E}{1}+4\binom{E}{2}, \quad E\ge 3. \tag{23}$$

Conjecture 4. *Row $n=6$:*

$$\sum_{E\ge 0} T(6,E)x^E = 13+11x+5x^2+x^3-\frac{13-43x+37x^2-8x^3}{(1-x)^4}. \tag{24}$$

$$T(6,E) = -13\binom{E}{0}+4\binom{E}{1}+10\binom{E}{2}+\binom{E}{3}, \quad E\ge 4. \tag{25}$$

Conjecture 5. *Row $n=7$:*

$$\sum_{E\ge 0} T(7,E)x^E = 12+20x+15x^2+6x^3+x^4-\frac{(4-3x)(3-5x-4x^2)}{(1-x)^4}. \tag{26}$$

$$T(7,E) = -12\binom{E}{0}-7\binom{E}{1}+23\binom{E}{2}+6\binom{E}{3}, \quad E\ge 5 \tag{27}$$

Conjecture 6. *Row $n=8$:*

$$\sum_{E\ge 0} T(8,E)x^E = -17+22x+31x^2+20x^3+7x^4+x^5+\frac{17-105x+257x^2-246x^3+78x^4}{(1-x)^5}. \tag{28}$$

$$T(8,E) = 17\binom{E}{0}-37\binom{E}{1}+44\binom{E}{2}+21\binom{E}{3}+\binom{E}{4}, \quad E\ge 6. \tag{29}$$

The question obviously left open here is how the properties of the linear diophantine equations can be translated to Transfer Matrices or Molien Invariants that emit such generating functions for the rows or columns without relying on adaptive fitting to finite numerical lists of solutions.

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