Maple-assisted proof of formula for A183821

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There are $4^6 = 4096$ configurations for a 2×3 sub-array, but we need only consider those where no 2×2 subblock is the reflection across the shared element pair of a horizontal neighbour.

We encode these configurations as lists in the order $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Thus the constraint is that $[x_1, x_4] \neq [x_3, x_6]$. This leaves 3840 configurations.

```
> Configs:= remove(t -> [t[1],t[4]]=[t[3],t[6]], [seq(convert(4^6+
  i,base,4)[1..6],i=0..4^6-1)]):
> nops(Configs);
```

$$3840 \quad \textbf{(1)}$$

Consider the 3840×3840 transition matrix T with entries $T_{ij} = 1$ if the first two rows of a 3×3 sub-array could be in configuration i while the last two rows are in configuration j, and 0 otherwise. The following code computes it.

```
> Compatible:= proc(i,j) local k;
   if Configs[i][4..6] <> Configs[j][1..3] or Configs[i][1..2] =
  Configs[j][4..5] or Configs[i][2..3] = Configs[j][5..6] then 0
  else 1 fi;
  end proc:
  T:= Matrix(3840,3840,Compatible):
```

Thus for $n \geq 1$ $\quad a(n) = \dfrac{u^T T^{n-1} u}{256}$ where u is a column vector of all 1's.

```
> u:= Vector(3840,1):
```

To check, here are the first few entries of our sequence.

```
> Tu[0]:= u:
  for n from 1 to 23 do Tu[n]:= T . Tu[n-1] od:
> A:= [seq(u^%T . Tu[n]/256,n=0..23)];
```

$A :=$ [15, 813, 43947, 2377341, 128578815, 6954559893, 376152548283, 20345104031589, 1100412240703119, 59518368554767389, 3219189893127084843, 174117402720326269485, 9417546276713264441151, 509369980225226329837125, 27550464744097268061031035, 1490131215317229298224019029, 80597226197051567988567042447, 4359289171265047916845288032141, 235782333652752792767026057120299, 12752838061125352602207353415866205, 689767024075645080785460202758062079, 37307660084895693807126550805828518773, 2017871908091487671585053292207288279739, 109141313826682286290748753877712780481733] **(2)**

Now here is the empirical recurrence formula. It says that $u^T T^n Q(T)\, u = 0$ for all nonnegative integers n, where Q is the following polynomial.

```
> n:= 'n': empirical:= a(n) = 43*a(n-1) + 645*a(n-2) - 2451*a(n-3):
```

```
Q:= unapply(add(coeff((lhs-rhs)(empirical),a(n-i))*t^(3-i),i=0..3),t);
```

$$Q := t \mapsto t^3 - 43\,t^2 - 645\,t + 2451 \tag{3}$$

But it suffices to show that $Q(T)\,u = 0$.

```
> QTu:= Tu[3] - 43 * Tu[2] - 645* Tu[1] + 2451 . Tu[0]:
  QTu^%T . QTu;
```

$$0 \tag{4}$$

This completes the proof.