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A181347/A181348, Logarithm of the Fibonacci matrix A037027

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The lower triangular matrix T, with rational elements $T(n,m)$:= A181347(n,m)/A181348(n,m), with offset $[0,0]$ is defined as $N \times N$ matrix for $N \geq 2$, with $T(0,0) = 0$, by:

$$\mathbf{T}_N := log(\mathbf{F}_N) \;, \tag{1}$$

with the NxN matrix from the first N rows and N columns of the lower triangular matrix $\mathbf{F}$:= A037027, the *Fibonacci* convolution matrix, which is a *Riordan* matrix of the *Bell* type $(Fib(x), x\, Fib(x))$, with $Fib(x) := \frac{1}{1-x-x^2}$ the *o.g.f.* of the *Fibonacci* sequence $[1,1,2,3,5,...]$.
Such a lower triangular *Riordan* matrix has diagonal elements $F(n,n) = 1$. This implies that the series for the matrix log terminates for every $N \geq 2$ because $(\mathbf{F}_N - \mathbf{1}_N)^N$ is the $N \times N$ $\mathbf{0}$-matrix ($\mathbf{1}_N$ is the $N \times N$ identity matrix). Therefore,

$$\log(\mathbf{F}_N) = -\sum_{k=1}^{N-1} \frac{(-1)^k}{k} (\mathbf{F}_N - \mathbf{1}_N)^k \;, \; N \geq 2 \;. \tag{2}$$

It is easy to collect the powers of $\mathbf{F}_N$ in this formula, and the result is

$$log(\mathbf{F}_N) = H_{N-1}\, \mathbf{1}_N + \sum_{p=1}^{N-1} \frac{(-1)^p}{p} \binom{N-1}{p} (\mathbf{F}_N)^p \,, \quad N \geq 2. \tag{3}$$

Here $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are the harmonic numbers A001008(n)/A002805(n).
In the derivation one uses the (easy to prove) identity

$$\sum_{k=p}^{N-1} \frac{1}{k} \binom{k}{k-p} = \frac{1}{p} \binom{N-1}{p} \,, \tag{4}$$

for $N \geq 2\,,\; p = 1, 2, ..., N-1$.
These two $\log(\mathbf{F}_N)$ formulae hold for the $N \times N$ submatrix of any Riordan matrix $\mathbf{F}_N := (G(x), x\,\hat{F}(x))$ with the o.g.f G of column $m = 0$ with $G(0) = 1$, and $\hat{F}(0) = 1$.

The first ten rows of the signed rational triangle $\mathbf{T}$ = A181347/A181348 are:

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TAB. 1: A181347(n,m)/A181348(n,m) signed, rational triangle

n/m	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	0								
2	1	2	0							
3	$-\frac{1}{2}$	2	3	0						
4	$-\frac{1}{3}$	-1	3	4	0					
5	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{3}{2}$	4	5	0				
6	$\frac{2}{15}$	$\frac{4}{3}$	-1	-2	5	6	0			
7	$-\frac{31}{30}$	$\frac{4}{15}$	2	$-\frac{4}{3}$	$-\frac{5}{2}$	6	7	0		
8	$-\frac{3}{70}$	$-\frac{31}{15}$	$\frac{2}{5}$	$\frac{8}{3}$	$-\frac{5}{3}$	-3	7	8	0	
9	$\frac{202}{105}$	$-\frac{3}{35}$	$-\frac{31}{10}$	$\frac{8}{15}$	$\frac{10}{3}$	-2	$-\frac{7}{2}$	8	9	0
⋮										

The corresponding matrix $log(\mathbf{F}_{10})$ looks like:

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$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & -1 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{2}{3} & -\frac{3}{2} & 4 & 5 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{15} & \frac{4}{3} & -1 & -2 & 5 & 6 & 0 & 0 & 0 & 0 \\ -\frac{31}{30} & \frac{4}{15} & 2 & -\frac{4}{3} & -\frac{5}{2} & 6 & 7 & 0 & 0 & 0 \\ -\frac{3}{70} & -\frac{31}{15} & \frac{2}{5} & \frac{8}{3} & -\frac{5}{3} & -3 & 7 & 8 & 0 & 0 \\ \frac{202}{105} & -\frac{3}{35} & -\frac{31}{10} & \frac{8}{15} & \frac{10}{3} & -2 & -\frac{7}{2} & 8 & 9 & 0 \end{bmatrix}$$

The row sums give the rational sequence (n=0..14):
$[0, 1, 3, \frac{9}{2}, \frac{17}{3}, \frac{15}{2}, \frac{142}{15}, \frac{152}{5}, \frac{2371}{210}, \frac{1481}{105}, \frac{3673}{210}, \frac{576}{35}, \frac{23341}{2310}, \frac{10085}{693}, \frac{344857}{606}]$. The numerator, resp. denominator sequences appear as A181349, resp. A181350.

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e.o.f.
