**Some simple continued fraction expansions for an infinite product, Part 2**

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## 1. Introduction

The infinite product

$$\Phi(a,x) = \prod_{n=0}^{\infty} \frac{1-ax^{4n+3}}{1-ax^{4n+1}}$$

converges for arbitrary complex $a$ provided $|x| < 1$. In the first part of these notes [1] we found the simple continued fraction expansion of $\Phi(a,x)$ when $x$ was a real algebraic number of the form

$$x = \frac{N\sqrt{m} - \sqrt{N^2m-4}}{2}$$

and $a$ was chosen equal to either $\sqrt{m}$ or $-\sqrt{m}$. Here $N$ and $m$ are positive integers such that $N^2m > 4$. The approach made use of a continued fraction expansion for $\Phi(a,x)$ due to Ramanujan.

In the second part of the notes we will find the simple continued fraction expansion of $\Phi(a,x)$ when $a$ is equal to either $i\sqrt{m}$ or $-i\sqrt{m}$ and $x$ is a purely imaginary algebraic number of the form

$$x = \frac{\sqrt{N^2m+4} - N\sqrt{m}}{2} i,$$

$N$ and $m$ being positive integers.

In Section 2 we give two general transformations that convert a continued fraction whose partial numerators are alternately $+1$ and $-1$ into a continued fraction with all partial numerators equal to $+1$. These transformations will be used in Section 3 to convert Ramanujan's continued fraction representation of $\Phi(a,x)$, for the particular values of $a$ and $x$ that we are considering, into the form of a simple continued fraction.

Our results were motivated by conjectures made by Paul Hanna in sequences A174504 through A174509, who considers the particular cases of the above corresponding to $m = 1$ and $N = 1, 2, 4, 6, 8$, or $N = 10$. Hanna works with the continued fraction expansion of a real number of the form

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n\left((-y)^n + y^{-n}\right)}\right)$$

with

$$y = \frac{\sqrt{N^2+4} - N}{2},$$

but it is not difficult to show that this real number is equal to $\Phi(-i, iy)$.

## 2. Some continued fraction transformations

In order to prove Lemma 2 we will need the following preliminary result (a variant of Proposition 1 from Part 1 of the notes).

**Proposition 3.** *If* $a_1, a_2, \ldots, a_{2n}$ *is a sequence of complex numbers then*

$$1+\frac{1}{a_1-1}+\frac{1}{a_2}-\frac{1}{a_3}+\frac{1}{a_4}-\cdots-\frac{1}{a_{2n-1}}+\frac{1}{a_{2n}} = \frac{1}{1}-\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots-\frac{1}{a_{2n-1}}+\frac{1}{a_{2n}}. \tag{1}$$

**Proof.** By induction on $n$. The result is easily verified for $n = 1$. Assume that equation (1) is true for a fixed integer $n > 1$. By an abuse of notation we let $F(n)$ denote the lhs of (1). Then we have

$$\begin{aligned}
F(n+1) &= 1+\frac{1}{a_1-1}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots-\frac{1}{a_{2n-1}}+\frac{1}{a_{2n}}-\frac{1}{a_{2n+1}}+\frac{1}{a_{2n+2}} \\
&= 1+\frac{1}{a_1-1}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots-\frac{1}{a_{2n-1}}+\frac{1}{\left(a_{2n}-\frac{1}{a_{2n+1}+\frac{1}{a_{2n+2}}}\right)} \\
&= \frac{1}{1}-\frac{1}{a_1}+\frac{1}{a_2}-\cdots-\frac{1}{a_{2n-1}}+\frac{1}{\left(a_{2n}-\frac{1}{a_{2n+1}+\frac{1}{a_{2n+2}}}\right)} \\
&= \frac{1}{1}-\frac{1}{a_1}+\frac{1}{a_2}-\cdots-\frac{1}{a_{2n-1}}+\frac{1}{a_{2n}}-\frac{1}{a_{2n+1}}+\frac{1}{a_{2n+2}},
\end{aligned}$$

where, in the penultimate step, we made use of the induction hypothesis. Thus the induction goes through and the proposition is proved. ■

In our next result we find two transformations that convert a continued fraction whose partial numerators are alternately $+1$ and $-1$ into a continued fraction with all partial numerators equal to $+1$.

**Lemma 2.** *If* $a_1, a_2, a_3, \ldots$ *is a sequence of complex numbers then*

(a)

$$\begin{aligned}
\frac{1}{1}-\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{a_3}+\cdots-\frac{1}{a_{2n-1}}+\frac{1}{a_{2n}} &= 1+\frac{1}{a_1-1}+\frac{1}{a_2-1}+\frac{1}{1}+\cdots \\
&\quad +\frac{1}{1}+\frac{1}{a_{2n-1}-1}+\frac{1}{a_{2n}-1}+\frac{1}{1}
\end{aligned}$$

(b)

$$\frac{1}{1+}\frac{1}{a_1-}\frac{1}{a_2+}\frac{1}{a_3-}\cdots-\frac{1}{a_{2n}+}\frac{1}{a_{2n+1}} = \frac{1}{1+}\frac{1}{a_1-1+}\frac{1}{1+}\frac{1}{a_2-1+}\frac{1}{a_3-1+}\frac{1}{1+}\cdots+\frac{1}{1+}\frac{1}{a_{2n}-1+}\frac{1}{a_{2n+1}-1+}\frac{1}{1}.$$

**Proof.** (a) The proof is by induction on $n$. The result is easily verified for $n = 1$. Assume that (a) is true for a fixed integer $n > 1$. By another abuse of notation we let $G(n)$ denote the rhs of (a). Then

$$\begin{aligned}
G(n+1) &= 1+\frac{1}{a_1-1+}\frac{1}{a_2-1+}\frac{1}{1+}\cdots+\frac{1}{1+}\frac{1}{a_{2n+1}-1+}\frac{1}{a_{2n+2}-1+}\frac{1}{1} \\
&= 1+\frac{1}{a_1-1+}\frac{1}{a_2-1+}\frac{1}{G(n)} \\
&= 1+\frac{1}{a_1-1+}\frac{1}{a_2-1+1-}\frac{1}{a_3+}\frac{1}{a_4-}\cdots-\frac{1}{a_{2n+1}+}\frac{1}{a_{2n+2}} \quad \text{(induction hypothesis)} \\
& \phantom{=} 1+\frac{1}{a_1-1+}\frac{1}{a_2-}\frac{1}{a_3+}\frac{1}{a_4+}\cdots-\frac{1}{a_{2n+1}+}\frac{1}{a_{2n+2}} \\
&= \frac{1}{1-}\frac{1}{a_1+}\frac{1}{a_2-}\frac{1}{a_3+}\cdots-\frac{1}{a_{2n+1}+}\frac{1}{a_{2n+2}},
\end{aligned}$$

by Proposition 3, and the proof by induction is complete.

(b) The result is an immediate consequence of (a). We have

$$\begin{aligned}
\frac{1}{1+}\frac{1}{a_1-}\frac{1}{a_2+}\frac{1}{a_3-}\cdots-\frac{1}{a_{2n}+}\frac{1}{a_{2n+1}} &= \frac{1}{1+}\frac{1}{a_1-1+1-}\frac{1}{a_2+}\frac{1}{a_3-}\cdots-\frac{1}{a_{2n}+}\frac{1}{a_{2n+1}} \\
&= \frac{1}{1+}\frac{1}{a_1-1+}\frac{1}{1+}\frac{1}{a_2-1+}\frac{1}{a_3-1+}\frac{1}{1+} \\
&\quad \cdots+\frac{1}{1+}\frac{1}{a_{2n}-1+}\frac{1}{a_{2n+1}-1+}\frac{1}{1}.
\end{aligned}$$

■

## 3. Simple continued fraction expansions

The following continued fraction expansion is a particular case of a more general result due to Ramanujan. For a proof consult [2, Entry 12 with $b = 0$ and $a^2$ replaced with $a$].

$$\Phi(a,x) = \prod_{n=0}^{\infty}\frac{1-ax^{4n+3}}{1-ax^{4n+1}} = \frac{1}{1-}\frac{ax}{1+x^2-}\frac{ax^3}{1+x^4-}\frac{ax^5}{1+x^6-}\cdots \qquad (2)$$

valid for arbitrary complex $a$ provided $|x| < 1$.

An equivalence transformation yields

$$\prod_{n=0}^{\infty} \frac{1-ax^{4n+3}}{1-ax^{4n+1}} = \frac{1}{1-} \frac{1}{\frac{1}{a}\left(\frac{1}{x}+x\right)-} \frac{1}{\frac{1}{x^2}+x^2-} \frac{1}{\frac{1}{a}\left(\frac{1}{x^3}+x^3\right)-} \frac{1}{\frac{1}{x^4}+x^4-} \cdots \tag{3}$$

valid for $0 < |x| < 1$.

There are several ways in which we can choose values for $a$ and $x$ so that the partial denominators of this continued fraction become integers. In Part 1 of these notes we considered two possible choices. We now consider two further cases.

**Case 1.**

Let $N$ and $m$ be positive integers and set $a = i\sqrt{m}$. Let $x_0$ denote the algebraic number

$$x_0 = \left\{ \frac{\sqrt{N^2m+4}-N\sqrt{m}}{2} \right\} i$$

so that $0 < |x_0| < 1$ and

$$x_0 + \frac{1}{x_0} = -iN\sqrt{m}. \tag{4}$$

A well-known property of $T_n(x)$, the $n$-th Chebyshev polynomial of the first kind, is the identity

$$T_n\left(\frac{x+x^{-1}}{2}\right) = \frac{x^n+x^{-n}}{2} \qquad x \neq 0.$$

Thus from equation (4)

$$x_0^n + \frac{1}{x_0^n} = 2T_n\left(-\frac{N\sqrt{m}}{2}i\right) \qquad n = 0, 1, 2, 3, ...$$

and the continued fraction expansion (3) becomes

$$\prod_{n=0}^{\infty} \frac{1-\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+3}}{1+\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+1}} = \frac{1}{1-} \frac{1}{\frac{-i}{\sqrt{m}}2T_1\left(-\frac{N\sqrt{m}}{2}i\right)-} \frac{1}{2T_2\left(-\frac{N\sqrt{m}}{2}i\right)-} \frac{1}{\frac{-i}{\sqrt{m}}2T_3\left(-\frac{N\sqrt{m}}{2}i\right)-} \frac{1}{2T_4\left(-\frac{N\sqrt{m}}{2}i\right)-} \cdots.$$

We now make an equivalence transformation so that the resulting partial numerators, after the first one, are alternately $+1$ and $-1$.

$$
\begin{aligned}
\prod_{n=0}^{\infty} \frac{1-\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+3}}{1+\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+1}} &= \frac{1}{1+}\,\frac{1}{\frac{i}{\sqrt{m}}2T_1\left(-\frac{N\sqrt{m}}{2}i\right)}-\frac{1}{-2T_2\left(-\frac{N\sqrt{m}}{2}i\right)} \\
&\quad +\frac{1}{\frac{-i}{\sqrt{m}}2T_3\left(-\frac{N\sqrt{m}}{2}i\right)}-\frac{1}{2T_4\left(-\frac{N\sqrt{m}}{2}i\right)} \\
&\quad +\frac{1}{\frac{i}{\sqrt{m}}2T_5\left(-\frac{N\sqrt{m}}{2}i\right)}-\frac{1}{-2T_6\left(-\frac{N\sqrt{m}}{2}i\right)} \\
&\quad +\frac{1}{\frac{-i}{\sqrt{m}}2T_7\left(-\frac{N\sqrt{m}}{2}i\right)}-\frac{1}{2T_8\left(-\frac{N\sqrt{m}}{2}i\right)}-\cdots \qquad (5) \\
&= \frac{1}{1+}\,\frac{1}{N-}\,\frac{1}{mN^2+2+}\,\frac{1}{mN^3+3N-} \\
&\quad \frac{1}{m^2N^4+4mN^2+2+}\,\frac{1}{m^2N^5+5mN^3+5N-}\cdots.
\end{aligned}
$$

Let us sketch the proof that the partial denominators $(-1)^n \frac{i}{\sqrt{m}} 2T_{2n+1}\left(-\frac{N\sqrt{m}}{2}i\right)$ and $(-1)^n 2T_{2n}\left(-\frac{N\sqrt{m}}{2}i\right)$ that occur in the expansion are positive integers. Firstly, one makes use of the recurrence equation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the Chebyshev polynomials to inductively prove that the partial denominators are integers. That they are positive integers then follows easily from the explicit formula

$$
\begin{aligned}
T_n(x) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k} \\
&= \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k} \quad (n > 0).
\end{aligned}
$$

We now apply Lemma 2 (b) to equation (5) to produce the expansion

$$\prod_{n=0}^{\infty} \frac{1-\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+3}}{1+\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+1}} = \frac{1}{1+}\frac{1}{\frac{i}{\sqrt{m}}2T_1\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\frac{1}{1+}\frac{1}{-2T_2\left(\frac{-N\sqrt{m}}{2}i\right)-1+}$$
$$\frac{1}{\frac{-i}{\sqrt{m}}2T_3\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\frac{1}{1+}\frac{1}{2T_4\left(-\frac{N\sqrt{m}}{2}i\right)-1+}$$
$$\frac{1}{\frac{i}{\sqrt{m}}2T_5\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\frac{1}{1+}\frac{1}{-2T_6\left(-\frac{N\sqrt{m}}{2}i\right)-1+}$$
$$\frac{1}{\frac{-i}{\sqrt{m}}2T_7\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\frac{1}{1+}\frac{1}{2T_8\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\cdots \quad (6)$$

$$= \frac{1}{1+}\frac{1}{N-1+}\frac{1}{1+}\frac{1}{mN^2+1+}\frac{1}{mN^3+3N-1+}\frac{1}{1+}$$
$$\frac{1}{m^2N^4+4mN^2+1+}\frac{1}{m^2N^5+5mN^3+5N-1+}\frac{1}{1+}\cdots$$

This is a simple continued fraction expansion for the infinite product except when $N = 1$.

In the case $N = 1$, equation (6) becomes the simple continued fraction expansion

$$\prod_{n=0}^{\infty} \frac{1-\sqrt{m}\left\{\frac{\sqrt{m+4}-\sqrt{m}}{2}\right\}^{4n+3}}{1+\sqrt{m}\left\{\frac{\sqrt{m+4}-\sqrt{m}}{2}\right\}^{4n+1}} = \frac{1}{2+}\frac{1}{-2T_2\left(\frac{-N\sqrt{m}}{2}i\right)-1+}\frac{1}{\frac{-i}{\sqrt{m}}2T_3\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\frac{1}{1+}$$
$$\frac{1}{2T_4\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\frac{1}{\frac{i}{\sqrt{m}}2T_5\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\frac{1}{1+}\cdots$$
$$= \frac{1}{2+}\frac{1}{m+1+}\frac{1}{m+2+}\frac{1}{1+}\frac{1}{m^2+4m+1+}$$
$$\frac{1}{m^2+5m+4+}\frac{1}{1+}\cdots. \quad (7)$$

We mention two specializations of these results. If in equation (7) we replace $m$

with $4m^2$ we get the simple continued fraction expansion

$$\prod_{n=0}^{\infty} \frac{1-2m\left\{\sqrt{m^2+1}-m\right\}^{4n+3}}{1+2m\left\{\sqrt{m^2+1}-m\right\}^{4n+1}} = \frac{1}{2+}\frac{1}{4m^2+1+}\frac{1}{4m^2+2+}\frac{1}{1+}\frac{1}{16m^4+16m^2+1+}\frac{1}{16m^4+20m^2+4+}\frac{1}{1+}\cdots,$$

whilst setting $N=2$ in equation (6) and replacing $m$ with $m^2$ yields the simple continued fraction expansion

$$\prod_{n=0}^{\infty} \frac{1-m\left\{\sqrt{m^2+1}-m\right\}^{4n+3}}{1+m\left\{\sqrt{m^2+1}-m\right\}^{4n+1}} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4m^2+1+}\frac{1}{8m^2+5+}\frac{1}{1+}\frac{1}{16m^4+16m^2+1+}\frac{1}{32m^4+40m^2+9+}\frac{1}{1+}\cdots.$$

**Case 2.**

Let $N$ and $m$ be positive integers but now we set $a=-i\sqrt{m}$. As before, we let $x_0$ denote the algebraic number

$$x_0 = \left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\} i$$

so that

$$x_0^n + \frac{1}{x_0^n} = 2T_n\left(-\frac{N\sqrt{m}}{2}i\right) \qquad n=0,1,2,3,....$$

The continued fraction expansion (3) becomes

$$\prod_{n=0}^{\infty} \frac{1+\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+3}}{1-\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+1}} = \frac{1}{1-}\frac{1}{\frac{i}{\sqrt{m}}2T_1\left(-\frac{N\sqrt{m}}{2}i\right)-}\frac{1}{2T_2\left(-\frac{N\sqrt{m}}{2}i\right)-}\frac{1}{\frac{i}{\sqrt{m}}2T_3\left(-\frac{N\sqrt{m}}{2}i\right)-}\frac{1}{2T_4\left(-\frac{N\sqrt{m}}{2}i\right)-}\cdots.$$

We use an equivalence transformation to arrange that the partial numerators are alternately $+1$ and $-1$.

$$\prod_{n=0}^{\infty} \frac{1+\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+3}}{1-\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+1}} = \frac{1}{1-}\,\frac{1}{\frac{i}{\sqrt{m}}2T_1\left(\frac{-N\sqrt{m}}{2}i\right)+}\,\frac{1}{-2T_2\left(-\frac{N\sqrt{m}}{2}i\right)}$$

$$-\frac{1}{\frac{-i}{\sqrt{m}}2T_3\left(-\frac{N\sqrt{m}}{2}i\right)+}\,\frac{1}{2T_4\left(-\frac{N\sqrt{m}}{2}i\right)}$$

$$-\frac{1}{\frac{i}{\sqrt{m}}2T_5\left(-\frac{N\sqrt{m}}{2}i\right)+}\,\frac{1}{-2T_6\left(-\frac{N\sqrt{m}}{2}i\right)}$$

$$-\frac{1}{\frac{-i}{\sqrt{m}}2T_7\left(-\frac{N\sqrt{m}}{2}i\right)+}\,\frac{1}{2T_8\left(-\frac{N\sqrt{m}}{2}i\right)-}\cdots$$

$$= \frac{1}{1-}\,\frac{1}{N+}\,\frac{1}{mN^2+2-}\,\frac{1}{mN^3+3N+}$$

$$\frac{1}{m^2N^4+4mN^2+2-}\,\frac{1}{m^2N^5+5mN^3+5N+}\cdots.$$

We now apply Lemma 2 (a) to give the expansion

$$\prod_{n=0}^{\infty} \frac{1+\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+3}}{1-\sqrt{m}\left\{\frac{\sqrt{N^2m+4}-N\sqrt{m}}{2}\right\}^{4n+1}} = 1+\frac{1}{\frac{i}{\sqrt{m}}2T_1\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\,\frac{1}{-2T_2\left(-\frac{N\sqrt{m}}{2}i\right)-1}$$

$$+\frac{1}{1+}\,\frac{1}{\frac{-i}{\sqrt{m}}2T_3\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\,\frac{1}{2T_4\left(-\frac{N\sqrt{m}}{2}i\right)-1}$$

$$+\frac{1}{1+}\,\frac{1}{\frac{i}{\sqrt{m}}2T_5\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\,\frac{1}{-2T_6\left(-\frac{N\sqrt{m}}{2}i\right)-1}$$

$$+\frac{1}{1+}\,\frac{1}{\frac{-i}{\sqrt{m}}2T_7\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\,\frac{1}{2T_8\left(-\frac{N\sqrt{m}}{2}i\right)-1+}\cdots$$

$$= 1+\frac{1}{N-1+}\,\frac{1}{mN^2+1+}\,\frac{1}{1+}\,\frac{1}{mN^3+3N-1+}$$

$$\frac{1}{m^2N^4+4mN^2+1+}\,\frac{1}{1+}\,\frac{1}{m^2N^5+5mN^3+5N-1+}\cdots. \tag{8}$$

Except for the case $N=1$, the partial denominators are all positive integers, so that equation (8) is the simple continued fraction expansion of the infinite product. Hanna has recorded the cases A174504 ($m=1,N=2$), A174506 ($m=1,N=4$), A174507 ($m=1,N=6$), A174508 ($m=1,N=8$) and A174509 ($m=1,N=10$).

In the case $N=1$, equation (8) becomes the simple continued fraction expansion

$$\prod_{n=0}^{\infty}\frac{1+\sqrt{m}\left\{\frac{\sqrt{m+4}-\sqrt{m}}{2}\right\}^{4n+3}}{1-\sqrt{m}\left\{\frac{\sqrt{m+4}-\sqrt{m}}{2}\right\}^{4n+1}} = m+2+\frac{1}{1+}\,\frac{1}{\frac{-i}{\sqrt{m}}2T_3\left(\frac{N\sqrt{m}}{2}i\right)-1+}\,\frac{1}{2T_4\left(\frac{N\sqrt{m}}{2}i\right)-1+}\,\frac{1}{1+}\cdots$$

$$= m+2+\frac{1}{1+}\,\frac{1}{m+2+}\,\frac{1}{m^2+4m+1+}\,\frac{1}{1+}\,\frac{1}{m^2+5m+4+}\cdots.$$

Hanna has recorded the case A174505 ($m=1$).

## REFERENCES


1. P. BALA, *Some simple continued fraction expansions for an infinite product, Part 1* (https://oeis.org/A174500/a174500_2.pdf)

2. B. C. BERNDT, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.