Some simple continued fraction expansions for an infinite product Part 1

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1. Introduction

The infinite product
$$\Phi(a,x) = \prod_{n=0}^{\infty} \frac{1-ax^{4n+3}}{1-ax^{4n+1}}$$
converges for arbitrary complex a provided $|x| < 1$. Let N and m be positive integers with $N^2m > 4$. Let x denote the real algebraic number
$$x = \frac{N\sqrt{m} - \sqrt{N^2m-4}}{2}.$$
We shall find the simple continued fraction expansion of the infinite products
$$\Phi(\sqrt{m},x) = \prod_{n=0}^{\infty} \frac{1-\sqrt{m}x^{4n+3}}{1-\sqrt{m}x^{4n+1}} \quad \text{and} \quad \Phi(-\sqrt{m},x) = \prod_{n=0}^{\infty} \frac{1+\sqrt{m}x^{4n+3}}{1+\sqrt{m}x^{4n+1}}$$
Our results were motivated by conjectures made by Paul Hanna in sequences A170540 through A170543, who considers the particular cases of the above corresponding to $m = 1$ and $N = 4, 6, 8$, or $N = 10$. Hanna works with the real number
$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n\left(x^n + x^{-n}\right)}\right)$$
but it is not difficult to show that this is equal to $\Phi(1,x)$.

2. Preliminaries on continued fractions

We adopt the standard compact notation
$$a_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots$$
to denote the general continued fraction
$$a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}. \tag{1}$$
We refer to the terms $a_n, n \geq 1$, in (1) as the *partial numerators* and the terms b_n as the *partial denominators* of the continued fraction. A *simple continued fraction* is a continued fraction in which a_0 is an integer, all the partial numerators are equal to 1 and each partial denominator is a positive integer. We recall (see for example [2, Theorem 14]) that every positive irrational real number has

a unique expansion as a simple continued fraction (with an infinite number of terms). Rational numbers have finite simple continued fraction expansions.
Given a sequence λ_n of non-zero complex numbers, the continued fraction

$$a_0 + \cfrac{\lambda_1 a_1}{\lambda_1 b_1 + \frac{\lambda_1\lambda_2 a_2}{\lambda_2 b_2 + \frac{\lambda_2\lambda_3 a_3}{\lambda_3 b_3 + \cdots}}} \quad (\lambda_n \neq 0) \tag{2}$$

is said to be obtained from (1) by means of an *equivalence transformation.* The continued fractions (1) and (2) are equivalent in the sense that the n-th convergents of both fractions have the same value for all n [3, p.19]. If in (1) the partial numerators a_n are all nonzero then we can choose complex numbers $\lambda_1, \lambda_2, \lambda_3$,... so that

$$1 = \lambda_1 a_1 = \lambda_1\lambda_2 a_2 = \lambda_2\lambda_3 a_3 = \dots \; .$$

By this means we can arrange that the partial numerators in the equivalent continued fraction (2) are all equal to 1.
In the next section we introduce another continued fraction transformation (Lemma 1) that converts a continued fraction with partial numerators equal to -1 into a continued fraction with all partial numerators equal to $+1$.

3. Some continued fraction transformations

In order to prove Lemma 1 we will need the following preliminary result.

Proposition 1. *If* $a_1, a_2, \dots, a_n$ *is a sequence of complex numbers then*

$$1 + \frac{1}{a_1 - 1 +} \frac{1}{a_2 +} \cdots \frac{1}{+ a_n} = \frac{1}{1 -} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots \frac{1}{+ a_n}. \tag{3}$$

Proof. By induction on n. The result is easily verified for $n = 1$. Assume that (3) is true for a fixed integer $n > 1$. Then by induction

$$\begin{aligned} 1 + \frac{1}{a_1 - 1 +} \frac{1}{a_2 +} \cdots \frac{1}{+ a_{n+1}} &= 1 + \frac{1}{a_1 - 1 +} \frac{1}{a_2 +} \cdots \frac{1}{+ a_{n-1} +} \frac{1}{\left(a_n + \frac{1}{a_{n+1}}\right)} \\ &= \frac{1}{1 -} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots \frac{1}{+ a_{n-1} +} \frac{1}{\left(a_n + \frac{1}{a_{n+1}}\right)} \\ &= \frac{1}{1 -} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots \frac{1}{+ a_{n+1}} \end{aligned}$$

and the induction goes through. ■

Lemma 1. *If* $a_1, a_2, ..., a_n$ *is a sequence of complex numbers then*

$$\frac{1}{1 -} \frac{1}{a_1 -} \frac{1}{a_2 -} \frac{1}{a_3 -} \cdots \frac{1}{- a_n} = 1 + \frac{1}{a_1 - 2 +} \frac{1}{1 +} \frac{1}{a_2 - 2 +} \frac{1}{1 +} \cdots \frac{1}{+ a_n - 2 +} \frac{1}{1}. \tag{4}$$

Proof. By induction on n. The result (4) is easily verified for $n = 1$. Assume that (4) is true for a fixed integer $n > 1$. Let $F(n)$ denote the rhs of (4). Then

$$\begin{aligned} F(n+1) &= 1 + \frac{1}{a_1 - 2} + \frac{1}{1} + \frac{1}{a_2 - 2} + \frac{1}{1} + \cdots + \frac{1}{a_n - 2} + \frac{1}{1} + \frac{1}{a_{n+1} - 2} + \frac{1}{1} \\ &= 1 + \frac{1}{a_1 - 2 + \frac{1}{A}}, \end{aligned}$$

where

$$\begin{aligned} A &= 1 + \frac{1}{a_2 - 2} + \frac{1}{1} + \cdots + \frac{1}{a_n - 2} + \frac{1}{1} + \frac{1}{a_{n+1} - 2} + \frac{1}{1} \\ &= \frac{1}{1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_{n+1}} \end{aligned}$$

by the induction hypothesis. Thus

$$\begin{aligned} F(n+1) &= 1 + \cfrac{1}{a_1 - 2 + 1 - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_{n+1}}} \\ &= 1 + \frac{1}{a_1 - 1} - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_{n+1}} \\ &= 1 + \frac{1}{a_1 - 1} + \frac{1}{(-a_2)} + \frac{1}{a_3} + \cdots + \frac{1}{((-1)^n a_{n+1})} \quad \text{[equivalence transformation]} \\ &= \frac{1}{1} - \frac{1}{a_1} + \frac{1}{(-a_2)} + \frac{1}{a_3} + \cdots + \frac{1}{((-1)^n a_{n+1})} \quad \text{[by Proposition 1]} \\ &= \frac{1}{1} - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_{n+1}}, \end{aligned}$$

where we have used another equivalence transformation to obtain the final expression. This completes the proof by induction. ■

4. Simple continued fraction expansions

The following continued fraction expansion is a particular case of a more general result due to Ramanujan. For a proof consult [1, Entry 12 with $b = 0$ and a^2 replaced with a].

$$\Phi(a, x) = \prod_{n=0}^{\infty} \frac{1 - ax^{4n+3}}{1 - ax^{4n+1}} = \frac{1}{1} - \frac{ax}{1 + x^2} - \frac{ax^3}{1 + x^4} - \frac{ax^5}{1 + x^6} - \cdots \tag{5}$$

valid for arbitrary complex a provided $|x| < 1$.
An equivalence transformation yields

$$\prod_{n=0}^{\infty} \frac{1 - ax^{4n+3}}{1 - ax^{4n+1}} = \frac{1}{1} - \frac{1}{\frac{1}{a}\left(\frac{1}{x} + x\right)} - \frac{1}{\frac{1}{x^2} + x^2} - \frac{1}{\frac{1}{a}\left(\frac{1}{x^3} + x^3\right)} - \frac{1}{\frac{1}{x^4} + x^4} - \cdots \tag{6}$$

valid for $0 < |x| < 1$.
There are several ways in which we can choose values for a and x so that the partial denominators of this continued fraction become integers. We deal with two cases here and leave consideration of a further two cases to Part 2 of these notes.

Case 1.

Let N and m be positive integers with $N^2m > 4$ and set $a = \sqrt{m}$. Let x_0 denote the real algebraic number

$$x_0 = \frac{N\sqrt{m} - \sqrt{N^2m-4}}{2}$$

so that $0 < x_0 < 1$ and

$$x_0 + \frac{1}{x_0} = N\sqrt{m}. \tag{7}$$

A well-known property of $T_n(x)$, the n-th Chebyshev polynomial of the first kind, is the identity

$$T_n\left(\frac{x+x^{-1}}{2}\right) = \frac{x^n + x^{-n}}{2} \qquad x \neq 0.$$

Thus from (7)

$$x_0^n + \frac{1}{x_0^n} = 2T_n\left(\frac{N\sqrt{m}}{2}\right) \qquad n = 0, 1, 2, 3, ...$$

and the continued fraction (6) becomes

$$\begin{aligned}\prod_{n=0}^{\infty} \frac{1-\sqrt{m}x_0^{4n+3}}{1-\sqrt{m}x_0^{4n+1}} &= \frac{1}{1-}\,\frac{1}{\frac{1}{\sqrt{m}}2T_1\left(\frac{N\sqrt{m}}{2}\right)-}\,\frac{1}{2T_2\left(\frac{N\sqrt{m}}{2}\right)-} \\ &\quad \frac{1}{\frac{1}{\sqrt{m}}2T_3\left(\frac{N\sqrt{m}}{2}\right)-}\,\frac{1}{2T_4\left(\frac{N\sqrt{m}}{2}\right)-}\cdots \\ &= \frac{1}{1-}\,\frac{1}{N-}\,\frac{1}{mN^2-2-}\,\frac{1}{mN^3-3N-} \\ &\quad \frac{1}{m^2N^4-4mN^2+2-}\cdots.\end{aligned}$$

Applying Lemma 1 to this continued fraction we obtain the continued fraction expansion

$$\begin{aligned}\prod_{n=0}^{\infty} \frac{1-\sqrt{m}x_0^{4n+3}}{1-\sqrt{m}x_0^{4n+1}} &= 1+\frac{1}{\frac{1}{\sqrt{m}}2T_1\left(\frac{N\sqrt{m}}{2}\right)-2+}\,\frac{1}{1+}\,\frac{1}{2T_2\left(\frac{N\sqrt{m}}{2}\right)-2} \\ &\quad +\frac{1}{1+}\,\frac{1}{\frac{1}{\sqrt{m}}2T_3\left(\frac{N\sqrt{m}}{2}\right)-2+}\,\frac{1}{1+}\,\frac{1}{2T_4\left(\frac{N\sqrt{m}}{2}\right)-2+}\cdots.\end{aligned} \tag{8}$$

Next we will show that the partial denominators of this continued fraction are positive integers, with three possible exceptions, so that, apart from these cases, (8) is the simple continued fraction representation of the real number given by the infinite product. In the case of the three exceptions we can still obtain a simple continued fraction expansion from (8) after a little extra work.

Proposition 2. *Let N, m be positive integers such that $N^2 m > 4$. Then for $k \geq 1$ both $2T_{2k}\left(\frac{N\sqrt{m}}{2}\right) - 2$ and $\frac{1}{\sqrt{m}} 2T_{2k+3}\left(\frac{N\sqrt{m}}{2}\right) - 2$ are positive integers.*

Proof. The Chebyshev polynomials $T_n(x)$ satisfy the recurrence equation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad [T_0(x) = 1, T_1(x) = x].$$

Calculation gives $2T_1\left(\frac{N\sqrt{m}}{2}\right) = N\sqrt{m}$ and $2T_2\left(\frac{N\sqrt{m}}{2}\right) = N^2m - 2$, an integer. A straightforward induction proof, making use of the recurrence equation, shows that the numbers $2T_{2k}\left(\frac{N\sqrt{m}}{2}\right)$ are integers, whilst the numbers $2T_{2k+1}\left(\frac{N\sqrt{m}}{2}\right)$ equal an integer multiplied by $\sqrt{m}$. Therefore the quantities $2T_{2k}\left(\frac{N\sqrt{m}}{2}\right) - 2$ and $\frac{1}{\sqrt{m}} 2T_{2k+3}\left(\frac{N\sqrt{m}}{2}\right) - 2$ are integers. We now show that they are positive integers greater than or equal to 3.
An explicit formula for the Chebyshev polynomials of the first kind is

$$T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (x^2 - 1)^k x^{n-2k}.$$

An easy consequence of this result is the inequality

$$T_n(x) > T_3(x) = 4x^3 - 3x \qquad [x > 1 \text{ and } n > 3]. \tag{9}$$

Thus for $k \geq 2$

$$\begin{aligned} 2T_{2k}\left(\frac{N\sqrt{m}}{2}\right) &> 2T_3\left(\frac{N\sqrt{m}}{2}\right) \\ &= N\sqrt{m}(N^2m - 3) \\ &\geq 2 \end{aligned}$$

if we recall that $N^2m > 4$.

In addition

$$\begin{aligned} 2T_2\left(\frac{N\sqrt{m}}{2}\right) &= N^2m - 2 \\ &> 2. \end{aligned}$$

Hence, for $k \geq 1$, $2T_{2k}\left(\frac{N\sqrt{m}}{2}\right) - 2$ is a positive integer.
Similarly, for $k \geq 1$ we have from (9)

$$\begin{aligned}
\frac{1}{\sqrt{m}} 2T_{2k+3}\left(\frac{N\sqrt{m}}{2}\right) &> \frac{1}{\sqrt{m}} 2T_3\left(\frac{N\sqrt{m}}{2}\right) \\
&= N(N^2m - 3) \\
&\geq 2.
\end{aligned}$$

and hence the partial denominator $\frac{1}{\sqrt{m}} 2T_{2k+3}\left(\frac{N\sqrt{m}}{2}\right) - 2$ is a positive integer. ■

The only partial denominators of the continued fraction (8) that are not positive integers are

$$\frac{1}{\sqrt{m}} 2T_1\left(\frac{N\sqrt{m}}{2}\right) - 2 = N - 2$$

when $N = 1$ or $N = 2$ and

$$\frac{1}{\sqrt{m}} 2T_3\left(\frac{N\sqrt{m}}{2}\right) - 2 = N(N^2m - 3) - 2$$

when $N = 1$ and $m = 5$ (recall the assumption $N^2m > 4$). In these cases we have to do a further simplification to get the continued fraction (8) into the form of a simple continued fraction. We state the final result in the form of a theorem.

Theorem 1. *Let N, m be positive integers such that $N^2m > 4$.*

(a) When $N \geq 3$ we have the simple continued fraction expansion

$$\begin{aligned}
\prod_{n=0}^{\infty} \frac{1 - \sqrt{m}\left(\frac{N\sqrt{m}-\sqrt{N^2m-4}}{2}\right)^{4n+3}}{1 - \sqrt{m}\left(\frac{N\sqrt{m}-\sqrt{N^2m-4}}{2}\right)^{4n+1}} &= 1 + \cfrac{1}{\frac{1}{\sqrt{m}}2T_1\left(\frac{N\sqrt{m}}{2}\right) - 2} + \frac{1}{1} + \cfrac{1}{2T_2\left(\frac{N\sqrt{m}}{2}\right) - 2} \\
&\quad + \frac{1}{1} + \cfrac{1}{\frac{1}{\sqrt{m}}2T_3\left(\frac{N\sqrt{m}}{2}\right) - 2} + \frac{1}{1} + \cfrac{1}{2T_4\left(\frac{N\sqrt{m}}{2}\right) - 2} + \cdots \\
&= 1 + \frac{1}{N-2} + \frac{1}{1} + \frac{1}{mN^2 - 4} + \frac{1}{1} + \frac{1}{mN^3 - 3N - 2} \\
&\quad + \frac{1}{1} + \frac{1}{mN^2(mN^2 - 4)} + \frac{1}{1} + \cdots.
\end{aligned}$$

(b) When $N = 2$ and $m > 1$ we have the simple continued fraction expansion

$$\begin{aligned}\prod_{n=0}^{\infty} \frac{1-\sqrt{m}\left(\sqrt{m}-\sqrt{m-1}\right)^{4n+3}}{1-\sqrt{m}\left(\sqrt{m}-\sqrt{m-1}\right)^{4n+1}} &= 2+\frac{1}{2T_2\left(\sqrt{m}\right)-2+}\,\frac{1}{1+}\,\frac{1}{\frac{1}{\sqrt{m}}2T_3\left(\sqrt{m}\right)-2+}\,\frac{1}{1+} \\ &\quad \frac{1}{2T_4\left(\sqrt{m}\right)-2+}\,\frac{1}{1+}\,\frac{1}{\frac{1}{\sqrt{m}}2T_5\left(\sqrt{m}\right)-2+}\,\frac{1}{1+}\cdots \\ &= 2+\frac{1}{4m-4+}\,\frac{1}{1+}\,\frac{1}{8m-8+}\,\frac{1}{1+}\,\frac{1}{16m^2-16m} \\ &\quad +\frac{1}{1+}\,\frac{1}{32m^2-40m+8+}\cdots.\end{aligned}$$

(c) When $N = 1$ and $m > 4$ we have the simple continued fraction expansion

$$\begin{aligned}\prod_{n=0}^{\infty} \frac{1-\sqrt{m}\left(\frac{\sqrt{m}-\sqrt{m-4}}{2}\right)^{4n+3}}{1-\sqrt{m}\left(\frac{\sqrt{m}-\sqrt{m-4}}{2}\right)^{4n+1}} &= -(m-3)+\frac{1}{m-4+}\,\frac{1}{1+}\,\frac{1}{2T_4\left(\frac{\sqrt{m}}{2}\right)-2+}\,\frac{1}{1} \\ &\quad +\frac{1}{\frac{1}{\sqrt{m}}2T_5\left(\frac{\sqrt{m}}{2}\right)-2+}\,\frac{1}{1+}\,\frac{1}{2T_6\left(\frac{\sqrt{m}}{2}\right)-2+}\,\frac{1}{1+}\cdots \\ &= -(m-3)+\frac{1}{m-4+}\,\frac{1}{1+}\,\frac{1}{m(m-4)+}\,\frac{1}{1+} \\ &\quad \frac{1}{m^2-5m+3+}\,\frac{1}{1+}\,\frac{1}{(m-4)(m-1)^2+}\cdots.\end{aligned}$$

■

Hanna has recorded four particular cases of part(a) of Theorem 1 in A174500 ($m = 1$ and $N = 4$), A174501 ($m = 1$ and $N = 6$), A174502 ($m = 1$ and $N = 8$) and A174503 ($m = 1$ and $N = 10$).

Case 2

We return to Ramanujan's continued fraction expansion (5). Let N and m be positive integers with $N^2m > 4$, as before, but now we set $a = -\sqrt{m}$. As before, let x_0 denote the real algebraic number

$$x_0 = \frac{N\sqrt{m}-\sqrt{N^2m-4}}{2}.$$

Then with these choices Ramanujan's continued fraction (5) becomes

$$\begin{aligned}
\Phi(-\sqrt{m}, x_0) &= \prod_{n=0}^{\infty} \frac{1+\sqrt{m}x_0^{4n+3}}{1+\sqrt{m}x_0^{4n+1}} \\
&= \frac{1}{1+}\,\frac{\sqrt{m}x}{1+x^2+}\,\frac{\sqrt{m}x^3}{1+x^4+}\,\frac{\sqrt{m}x^5}{1+x^6+}\cdots \\
&= \frac{1}{1+}\,\frac{1}{\frac{1}{\sqrt{m}}\left(\frac{1}{x}+x\right)+}\,\frac{1}{\frac{1}{x^2}+x^2+}\,\frac{1}{\frac{1}{\sqrt{m}}\left(\frac{1}{x^3}+x^3\right)+}\,\frac{1}{\frac{1}{x^4}+x^4+}\cdots
\end{aligned}$$

by an equivalence transformation. Expressing the partial denominators of this continued fraction in terms of the Chebyshev polynomials of the first kind we have the following result.

Theorem 2. *Let N, m be positive integers such that $N^2m > 4$.* There holds the simple continued fraction expansion

$$\begin{aligned}
\prod_{n=0}^{\infty} \frac{1+\sqrt{m}\left(\frac{N\sqrt{m}-\sqrt{N^2m-4}}{2}\right)^{4n+3}}{1+\sqrt{m}\left(\frac{N\sqrt{m}-\sqrt{N^2m-4}}{2}\right)^{4n+1}} &= \frac{1}{1+}\,\frac{1}{\frac{1}{\sqrt{m}}2T_1\left(\frac{N\sqrt{m}}{2}\right)+}\,\frac{1}{2T_2\left(\frac{N\sqrt{m}}{2}\right)+} \\
&\quad \frac{1}{\frac{1}{\sqrt{m}}2T_3\left(\frac{N\sqrt{m}}{2}\right)+}\,\frac{1}{2T_4\left(\frac{N\sqrt{m}}{2}\right)+}\cdots \\
&= \frac{1}{1+}\,\frac{1}{N+}\,\frac{1}{mN^2-2+}\,\frac{1}{mN^3-3N+} \\
&\quad \frac{1}{m^2N^4-4mN^2+2+\cdots}.
\end{aligned}$$

■

In the particular case $m = 1$, the sequence of partial denominators of this simple continued fraction becomes the sequence $[1, 2T_1\left(\frac{N}{2}\right), 2T_2\left(\frac{N}{2}\right), 2T_3\left(\frac{N}{2}\right), \ldots]$. There are many sequences of this type currently in the OEIS database (but with initial term 2 rather than 1): see A005248 ($N = 3$), A003500 ($N = 4$), A003501 ($N = 5$), A003499 ($N = 6$), A056854 ($N = 7$), A086903 ($N = 8$), A056918 ($N = 9$), A087799 ($N = 10$), A057076 ($N = 11$), A087800 ($N = 12$), A078363 ($N = 13$), A067902 ($N = 14$), A078365 ($N = 15$), A090727 ($N = 16$), A078367 ($N = 17$), A087215 ($N = 18$), A078369 ($N = 19$), A090728 ($N = 20$), A090729 ($N = 21$), A090730 ($N = 22$), A090731 ($N = 23$), A090732 ($N = 24$), A090733 ($N = 25$), A090247 ($N = 26$), A090248 ($N = 27$), A090249 ($N = 28$) and A090251 ($N = 29$).

In Part 2 of theses notes we find the simple continued fraction expansions of the infinite products

$$\prod_{n=0}^{\infty} \frac{1-\sqrt{m}x^{4n+3}}{1+\sqrt{m}x^{4n+1}} \quad \text{and} \quad \prod_{n=0}^{\infty} \frac{1+\sqrt{m}x^{4n+3}}{1-\sqrt{m}x^{4n+1}},$$

where now x denotes an algebraic number of the form

$$x = \frac{\sqrt{N^2m+4} - N\sqrt{m}}{2}.$$

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