A168624 and some empirical continued fraction expansions

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The terms of A168624 are given by

$$a(n) \quad = \quad 10^{2n} - 10^n + 1.$$

We make some empirical observations about patterns in the simple continued fraction expansion of the quadratic irrationals $\sqrt{a(n)}$.

A related sequence A066138, is defined by the formula

$$b(n) \quad = \quad 10^{2n} + 10^n + 1.$$

The simple continued fraction expansion of $\sqrt{b(n)}$ has period 6

$$\sqrt{b(n)} = [10^n; \overline{1, 1, \frac{2(10^n - 1)}{3}, 1, 1, 2 \times 10^n}, ...].$$

If we examine Table 1 we see there is no such simple result for the continued fraction expansion of $\sqrt{a(n)}$.

Table 1.

n	Continued fraction expansion of $\sqrt{10^{2n} - 10^n + 1}$
2	[99; 1, 1, 65, 1, 5, 21, 1, 17, 7, 3, 5, 1, 2, 2, ...]
3	[999; 1, 1, 665, 1, 5, 221, 1, 17, 73, 1, 53, 24, 1, 1, ...]
4	[9999; 1, 1, 6665, 1, 5, 2221, 1, 17, 740, 1, 1, 1, 5, 2, ...]
5	[99999; 1, 1, 66665, 1, 5, 22221, 1, 17, 7407, 3, 5, 1,2, 2468, ...]
6	[999999; 1, 1, 666665, 1, 5, 222221, 1, 17, 74073, 1, 53, 24691, 3, 17, ...]
7	[9999999; 1, 1, 6666665, 1, 5, 2222221, 1, 17, 740740, 1, 1, 1, 5, 2, ...]

However, we do see some structure in these expansions: we notice that the continued fraction expansion of $\sqrt{a(n)}$, for $n \ge 2$, apparently always begins

$$\sqrt{a(n)} \quad = \quad [L; 1, 1, L, 1, 5, L, 1, 17, L, ...]. \qquad (1)$$

Here L denotes a partial quotient that depends on n (and its postion in the expansion), and which appears to increase with n in a predictable manner.

The next few terms in the expansion (1) seem to depend on the value of n modulo 3. Further calculation suggests we have the expansions for $n \ge 1$

$$\sqrt{a(3n)} = [L;1,1,L,1,5,L,1,17,L,1,53,L,...]$$

$$\sqrt{a(9n)} = [L;1,1,L,1,5,L,1,17,L,1,53,L,1,161,L,...]$$

$$\sqrt{a(27n)} = [L;1,1,L,1,5,L,1,17,L,1,53,L,1,161,L,1,485,L,...]$$

and so on. The sequence of small partial quotients 1, 5, 17, 53, 161, 485, ... is presumably A048473(n) $= 2 \times 3^n - 1$.

Empirically, we see the following structure in the initial terms of the continued fraction expansions for $n \geq 1$

$$\sqrt{a(3n+1)} = [L;1,1,L,1,5,L,1,17,L,1,1,1,5,2,1,L,...]$$

$$\sqrt{a(3(3n+1))} = [L;1,1,L,1,5,L,1,17,L,1,53,L,1,1,1,17,2,1,L,...]$$

$$\sqrt{a(9(3n+1))} = [L;1,1,L,1,5,L,1,17,L,1,53,L,1,161,L,1,1,1,53,2,1,L,...]$$

$$\sqrt{a(27(3n+1))} = [L;1,1,L,1,5,L,1,17,L,1,53,L,1,161,L,1,485,L,1,1,1,161,2,1,L,...]$$

and so on.

We see similar structure in the initial terms of the continued fraction expansions of numbers of the form $\sqrt{a(3^k(3n+2))}$, $k = 0,1,2,....$

We also have similar conjectural behaviour for the continued fraction expansions of multiples of $\sqrt{a(n)}$. For example, as k increases, the simple continued fraction expansion of the numbers $2\sqrt{a(3^k n)}$ conjecturally has the initial structure

$$[L;L,1,1,L,1,2,L,1,8,L,1,26,L,1,242,L,1,728,L,...],$$

with large partial quotients L depending on n. The sequence of small partial quotients 2, 8, 26, 242, 728, ... is presumably A024023(n) $= 3^n - 1$. Of course, it would be nice to have proofs of these conjectures.