NOTES ON SEQUENCES A145502 - A145510

Peter Bala, November 12 2012

Let $x > 2$ and define a sequence $\{a(n)\}_{n\geq 1}$ by setting $a(1) = x - 1$ and for $n > 1$ setting

$$a(n) = \left(\frac{x+\sqrt{x^2-4}}{2}\right)^{2^{n-1}} + \left(\frac{x-\sqrt{x^2-4}}{2}\right)^{2^{n}-1} - 1.$$

The dependence of $a(n)$ on x is suppressed for notational convenience. It is easy to verify that $a(n)$ satisfies the recurrence equation $a(n+1) = a(n)^2 + 2a(n) - 2$ with the initial condition $a(1) = x - 1$. Sequences A145502 through A145510 correspond to the cases $x = 3$ through $x = 11$ respectively.

A product expansion

These sequences can be used to give rapidly converging product expansions to evaluate quadratic irrationalities. We shall prove the following identity, valid for $x > 2$:

$$\frac{x+1}{\sqrt{x^2-4}} = \prod_{n=1}^{\infty}\left(1+\frac{1}{a(n)}\right). \tag{1}$$

This result follows immediately on taking the limit in the finite product expansion

$$\frac{x+1}{\sqrt{x^2-4}} = \frac{a(n+1)+2}{\sqrt{a(n+1)^2+2a(n+1)-3}}\prod_{k=1}^{n}\left(1+\frac{1}{a(k)}\right). \tag{2}$$

We give an inductive proof of this latter identity. Let $P(n)$ denote the rhs of (2),

$$P(n) = \frac{a(n+1)+2}{\sqrt{a(n+1)^2+2a(n+1)-3}}\prod_{k=1}^{n}\left(1+\frac{1}{a(k)}\right). \tag{3}$$

The initial value is

$$P(0) = \frac{a(1)+2}{\sqrt{a(1)^2+2a(1)-3}} = \frac{x+1}{\sqrt{x^2-4}}.$$

We shall prove $P(n) = P(0)$ is a constant, independent of n, by proving that the ratio $P(n+1)/P(n) = 1$. The proof makes repeated use of the recurrence equation $a(n+1) = a(n)^2 + 2a(n) - 2$.

From the definition (3) of $P(n)$ we find

$$\begin{aligned}
\frac{P(n+1)}{P(n)} &= \frac{a(n+2)+2}{\sqrt{a(n+2)^2+2a(n+2)-3}} \frac{\sqrt{a(n+1)^2+2a(n+1)-3}}{a(n+1)+2} \frac{a(n+1)+1}{a(n+1)} \\
&= \frac{1}{\sqrt{a(n+2)^2+2a(n+2)-3}} \frac{\sqrt{a(n+1)^2+2a(n+1)-3}}{1} \frac{a(n+1)+1}{1} \\
&= \frac{\sqrt{a(n+1)^2+2a(n+1)-3}}{\sqrt{(a(n+2)+3)(a(n+2)-1)}}(a(n+1)+1) \\
&= \frac{\sqrt{a(n+1)^2+2a(n+1)-3}}{\sqrt{(a(n+1)+1)^2(a(n+2)-1)}}(a(n+1)+1) \\
&= \frac{\sqrt{a(n+1)^2+2a(n+1)-3}}{\sqrt{(a(n+2)-1)}} \\
&= 1
\end{aligned}$$

Hence by induction $P(n) = P(0) = (x+1)/\left(\sqrt{x^2-4}\right)$ is true for all n and (1) is proved.

A seond product expansion

In an exactly similar manner to the above we can establish the product expansion

$$\sqrt{\frac{x+2}{x-2}} = \prod_{n=1}^{\infty}\left(1+\frac{2}{a(n)+1}\right) \qquad [x = 1 + a(1) > 2], \tag{4}$$

(Cantor [2], but which apparently dates back to Euler [1]) and its finite counterpart

$$\sqrt{\frac{x+2}{x-2}} = \sqrt{\frac{a(n+1)+3}{a(n+1)-1}} \prod_{k=1}^{n}\left(1+\frac{2}{a(k)+1}\right) \qquad [x = 1 + a(1) > 2]. \tag{5}$$

Related sequences

Put $b(n) = a(n) + 1$. The sequence $b(n)$ satisifes the recurrence $b(n+1) = b(n)^2 - 2$, with the initial condition $b(1) = x$. Currently in the database are the sequences A001566 ($x = 3$), A003010 ($x = 4$), A003487 ($x = 5$) and A003423 ($x = 6$).
Since

$$\left(1 - \frac{1}{b(n)}\right) = \frac{1}{\left(1+\frac{1}{a(n)}\right)},$$

identity (1) when expressed in terms of the b-sequences takes the form

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{b(n)}\right)=\frac{\sqrt{x^2-4}}{x+1}, \qquad [x=b(1)>2].$$

For example, when $x=4$ we have

$$\frac{2}{5}\sqrt{3}=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{14}\right)\left(1-\frac{1}{194}\right)\left(1-\frac{1}{37634}\right)...,$$

where $4, 14, 194, 37634, ...$ is the Lucas-Lehmer sequence A003010.

Identity (4) when expressed in terms of the b-sequences takes the form

$$\prod_{n=1}^{\infty}\left(1+\frac{2}{b(n)}\right)=\sqrt{\frac{x+2}{x-2}}, \qquad [\ x=b(1)>2].$$

For example, when $x=3$ we have

$$\sqrt{5}=\left(1+\frac{2}{3}\right)\left(1+\frac{2}{7}\right)\left(1+\frac{2}{47}\right)\left(1+\frac{2}{2207}\right)...,$$

where $3, 7, 47, 2207, ...$ is A001566.

Put $c(n)=\frac{1}{2}(a(n)+1)$. The sequence $c(n)$ satisifes the recurrence $c(n+1)=2c(n)^2-1$, with the initial condition $c(1)=x/2=X$. Currently in the database are the sequences A002812 ($X=2$), A001601 ($X=3$), A005828 ($X=4$) and A084765 ($X=5$). The product identities (1) and (4) for these sequences become

$$\frac{\sqrt{4X^2-4}}{2X+1}=\prod_{n=1}^{\infty}\left(1-\frac{1}{2c(n)}\right) \qquad [X=c(1)]. \tag{6}$$

and

$$\sqrt{\frac{X+1}{X-1}}=\prod_{n=1}^{\infty}\left(1+\frac{1}{c(n)}\right) \qquad [X=c(1)]. \tag{7}$$

For further product expansion results see A209010.

REFERENCES

[1] M. Mendes France and A. J. van der Poorten, From geometry to Euler identities, Theoret. Comput. Sci., 65 (1989), 213-220.

[2] G. Cantor Gesammelte Abhandlungen, p.43