The C_4 lattice and a continued fraction for log(2)

Peter Bala, March 14 2024

A142993: the crystal ball sequence for the lattice C_4.

A142993(n) := P(n) is the polynomial function

$$P(n) = (2n + 1)^2 (4n^2 + 4n + 3)/3.$$

A CAS such as Maple can be used to evaluate the series

$$\sum_{k = 0..n} 1/((k + 1) P(k) P(k+1)) = 17/12 - 2\log(2).$$

The purpose of this note is to convert the series to a continued fraction.

Recall the fundamental 3-term recurrences for the numerator and denominator of a continued fraction. If we write the finite continued fraction

$$R(n) = a(1)/(b(1) + a(2)/(b(2) + \dots + a(n)/(b(n))))$$

in the form

$$R(n) = A(n)/B(n)$$

then A(n) and B(n) are polynomials in a(i), b(j), that, for n >= 2, satisfy the 3-term recurrences

$$A(n) = b(n) A(n-1) + a(n) A(n-2)$$

$$B(n) = b(n) B(n-1) + a(n) B(n-2)$$

with initial values

$$A(1)/B(1) = a(1)/b(1)$$

and

$$A(2)/B(2) = a(1) b(2)/(b(1) b(2) + a(2)).$$

Returning to A142993, we define sequences {A(n) : n >= 0} and {B(n) : n > = 0} by

$$A(n) = B(n) \sum_{k = 0..n} 1/((k + 1) P(k) P(k+1))$$

$$B(n) = P(n+1) (2n + 2)!,$$

so that A(n)/B(n) = Sum_{k = 0..n} 1/((k + 1)*P(k)*P(k+1)).

The sequence {B(n)} is clearly integral; it will turn out that {A(n)} is also integral. The first few values are

n	0	1	2	3
A(n)	2	164	18216	2744352
B(n)	66	5400	599760	90357120

We show that {A(n)} and {B(n)} satisfy the same 3-term recurrence.

Firstly, it is easy to check that B(n) satisfies the 3-term recurrence

u(n) = 2*(4*n^2 + 4*n + 33)*u(n-1) - 4*n^2*(4*n^2 - 1)*u(n-2).

We show that A(n) satisfies the same recurrence (thus showing that A(n) is an integer). By definition

A(n) = B(n)*Sum_{k = 0..n} 1/((k + 1)*P(k)*P(k+1)).

Hence

A(n+1) = B(n+1)*Sum_{k = 0..n+1} 1/((k + 1)*P(k)*P(k+1))

= B(n+1)*Sum_{k = 0..n} 1/((k + 1)*P(k)*P(k+1))

+ B(n+1)/((n + 2)*P(n+1)*P(n+2))

= (B(n+1)/B(n))*A(n) + B(n+1)/((n + 2)*P(n+1)*P(n+2))

Substituting B(n) = P(n+1)*(2*n+2)! and multiplying both sides of the resulting identity by (n+2)*P(n+1) we find that

(n+2)*P(n+1)*A(n+1)= (n+2)*(2*n+3)*(2*n+4)*P(n+2)*A(n)

+ (2*n + 4)! ... (1)

Hence

(n+3)*P(n+2)*A(n+2)= (n+3)*(2*n+5)*(2*n+6)*P(n+3)*A(n+1)

+ (2*n + 6)! ... (2)

Multiplying (1) by (2*n + 5)*(2*n + 6) and subtracting from (2) and then replacing n with n - 2 we find after a short calculation that A(n) satisfies the same 3-term recurrence as satisfied by B(n):

$$A(n) = 2*(4*n^2 + 4*n + 33)*A(n-1) - 4*n^2*(4*n^2 - 1)*A(n-2).$$

The first few coefficients of the recurrence are shown below.

n	1	2	3	4
4*n^2*(4*n^2 - 1)	12	240	1260	4032
2*(4*n^2 + 4*n + 33)	82	114	162	226

It then follows from the fundamental recurrences satisfied by the numerators and denominators of a continued fraction that the series

$$\text{Sum}_{k >= 0} 1/((k + 1)*P(k)*P(k+1))$$

$$= \text{Limit}_{n \to oo} A(n)/B(n)$$

has the continued fraction expansion

$$\cfrac{2}{66 - \cfrac{12}{82 - \cfrac{240}{114 - \cfrac{1260}{162 - \dots}}}}$$

By means of an equivalence transformation this is equal to the continued fraction

$$\cfrac{1}{33 - \cfrac{3}{41 - \cfrac{60}{57 - \dots}}}$$

with partial numerators and partial denominators (after the first) equal to n^2*(4*n^2 - 1) and 2*(4*n^2 + 4*n + 33) (= 2*((2*n + 1)^2 + 2*(4^2))) respectively.

A similar result holds for the crystal ball sequences of the other C_n lattices.