Why A139669(n) is not generalisable to all $p = 4k + 3$ where $p + 1$ is not a power of 2, and a correct generalisation.

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(1) The number of isomorphism classes of order $2^n \cdot 11$ is not the least possible for $n \geq 10$

For $0 \leq n \leq 9$, every group G of order $2^n \cdot 11$ contains C_{11} as a unique Sylow subgroup. Its uniqueness is a consequence of the Sylow theorems: let n_{11} be the number of Sylow 11-subgroups; then n_{11} divides 2^n, and $n_{11} \equiv 1 \pmod{11}$. Because the minimum 2^m such that $2^m \equiv 1 \pmod{p}$ is 2^{10}, when $0 \leq n \leq 9$ and thus $2^{10} > 2^n$, $n_{11} \equiv 1 \pmod{11}$ only when $n_{11} = 1$. Therefore C_{11} is unique and normal for $0 \leq n \leq 9$, and by the Schur-Zassenhaus theorem, every G of order $2^n \cdot 11$ is a semidirect product $C_{11} \rtimes H$, where H is an arbitrary 2-group of order 2^n. The number of isomorphism classes is thus the number of semidirect products $C_{11} \rtimes H$, which is equivalent to the number of homomorphisms $\sigma : H \to \mathrm{Aut}(C_{11}) \cong C_{10}$. (Which, since H is a 2-group and $|C_{10}| = 2 \cdot 5$, there are only two possible: σ_1, where $\mathrm{im}(\sigma_1) \cong H/N \cong \{e\}$; and σ_2, where $\mathrm{im}(\sigma_2) \cong H/N \cong C_2$.)

But when $n = 10$, C_{11} is no longer necessarily unique: $n_{11} \equiv 1 \pmod{11}$ is also satisfied when $n_{11} = 2^{10}$. Consequently, there exist extra groups of order $2^{10} \cdot 11$ isomorphic to the semidirect product $H \rtimes C_{11}$, where H is now unique and normal, and C_{11} is not. And since all semidirect products $C_{11} \rtimes H$ will still exist, any $H \rtimes C_{11}$ will be additional, and the number of groups will be greater. This applies for all $n \geq 10$. (In fact we can construct one easily: $E_{2^{10}} \rtimes C_{11}$, where $E_{2^{10}}$ is the elementary abelian of order 2^{10}, with homomorphism $\sigma : C_{11} \to \mathrm{Aut}(E_{2^{10}})$; $|\mathrm{Aut}(E_{2^{10}})| = 2^{45} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127$, hence $C_{11} \subset \mathrm{Aut}(E_{2^{10}})$, and $\mathrm{im}(\sigma) = C_{11}$, making σ injective. And since 11 is a factor in $|\mathrm{Aut}(E_{2^n})|$ for $n \geq 10$ (for $n!$ divides $|\mathrm{Aut}(E_{2^n})|$ since the symmetric group S_n is a subgroup) therefore a similar homomorphism exists for all E_{2^n}.)

(2) A139669(n) is not generalisable to all $p = 4k+3$ where $p+1$ is not a power of 2

Since there exists such a minimum 2^m for every p (Fermat's little theorem), the number of isomorphism classes will eventually differ for each p. For example, for $p = 3$, the number of isomorphism classes of order $2^2 \cdot 3$ is 5 instead of $4 =$ A139669(2), because of the alternating group on 4 elements A_4, which is isomorphic to $(C_2 \times C_2) \rtimes C_3$. And for $p = 31$, the number of isomorphism classes is 196 instead of $195 =$ A139669(5), because of the group $E_{2^5} \rtimes C_{31}$, where E_{2^5} is the elementary abelian of order 2^5. Although these both agree with the initial comment in A139669, we can expect similar differences for $p = 19$ at $n = 18$, $p = 23$ at $n = 11$, and more generally for $p =$ A000040(A080148(n)) at $n =$ A014664(A080148(n)).

(3) A correct generalisation

For every $p = 4k + 3$, the number of isomorphism classes of order $2^n p$ will be the same for every such p with a minimum $2^m \equiv 1 \pmod{p}$ such that $2^m > 2^n$. This is because counting the number of groups reduces to counting the number of semidirect products $C_p \rtimes H$, which is equivalent to counting the number of homomorphisms $\sigma : H \to \mathrm{Aut}(C_p) \cong C_{p-1}$. The condition of minimum $2^m \equiv 1 \pmod{p}$ such that $2^m > 2^n$ ensures that C_p is always normal and $C_p \rtimes H$ is the only semidirect product (for reasons stated in **(1)**). Crucially, since the existence of the homomorphism depends on $p - 1$ and not

p, and since H is a 2-group, only the power of 2 in $p-1$ determines the existence of a homomorphism. Since for $p = 4k+3$, $p-1 = 2r$, with r odd, the number of homomorphisms is the same for all such p. (See the Miles Englezou link in A376349 for a similar proof relating to another similar sequence).

(4) 1, 2, 4, 12, 42, 195, 1387, 19324, 1083472, ... are the least possible number of groups of order $2^n p$

For $p = 4k+3$ satisfying $2^m > 2^n$ as defined above, the number of isomorphism classes for $|G| = 2^n p$ is the smallest possible. As we have established we are restricted to semidirect products $C_p \rtimes H$. Since $p = 4k+3$, 2 is the maximum power of 2 dividing $p-1$. This means that for the homomorphism $\sigma : H \to \mathrm{Aut}(C_p) \cong C_{p-1}$ there are only two possible: σ_1, where $\mathrm{im}(\sigma_1) \cong H/N \cong \{e\}$; and σ_2, where $\mathrm{im}(\sigma_2) \cong H/N \cong C_2$. Let q be a prime such that 2^k is the maximum power of 2 dividing $q-1$, $k > 1$. Hence now we may have additional homomorphisms σ_i for $1 < i \leq k$, where $\mathrm{im}(\sigma_i) \cong H/N \cong C_{2^i}$. But since $|H| = 2^n$ and every 2-group contains C_2 as a subgroup, it follows that every homomorphism that exists for p will exist for q. Since 2^1 is the least possible power of 2 dividing $p-1$ for arbitrary p, therefore when $p = 4k+3$, the number of isomorphism classes for groups of order $2^n p$ is the least possible.