Entanglement Permutations

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(*This July 27 2016 draft with slight improvements as suggested by Neil Sloane and anonymous referees of Bridges 2016 conference*)

Abstract

We present a generic technique for creating permutations of natural numbers with interesting properties. Many of these permutations seem to have potential for mathematical art, in both audio and visual realms.

Introduction

I have for some years searched for instances of integer sequences (see Sloane [1] for the ever-growing collection) that are neither overly regular nor too symmetric (and thus *boring*), or so chaotic or rhythmless that it is hard to discern any structure at all. After seeing the still image of OEIS-movie [2], and finding that the still image was obtained from the scatter plot of Katarzyna Matylla's sequence A135141 (Figure 1, all A-numbers refer to sequences in [1]), I started forming similar recurrences in a systematic manner. Sometimes these matched with existing sequences, and in the majority of cases, the resulting scatter plot had interesting scale-invariant structure.

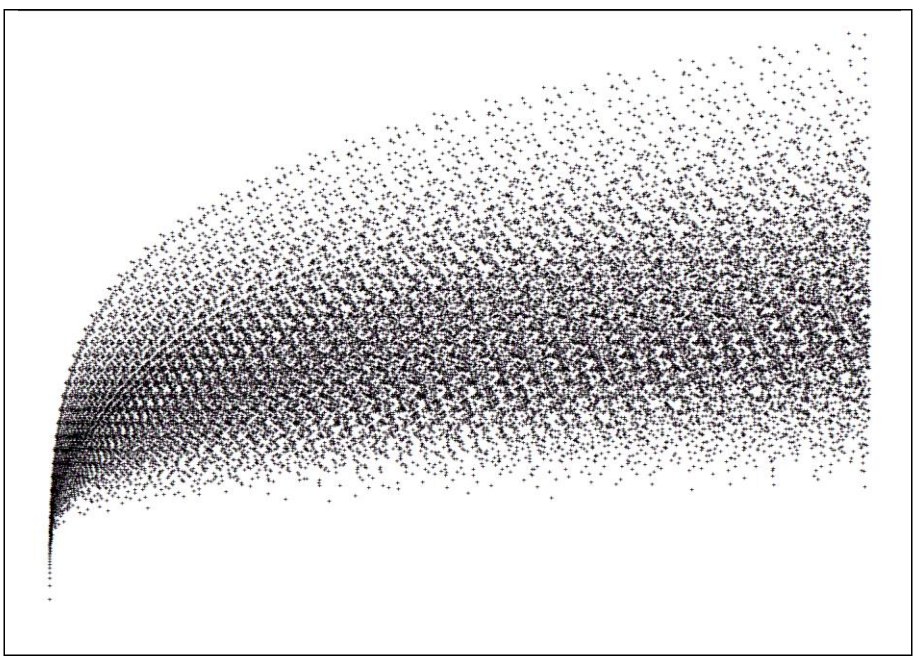

Figure 1: *Logarithmic scatter plot of Matylla's A135141: (primes, composites) ⤩ (even, odd).*

Formal Definition

We use four letters from Ethiopic syllabary [3] as meta-variable names for integer sequences, for their easy memorability and pronounceability: ሌ [le], ሉ [lu], ሔ [he] and ሑ [hu]. Let (ሌ, ሉ) and (ሔ, ሑ) be two ordered pairs of complementary sequences in set $\mathbb{N} \setminus \{1\}$. In other words, sequences ሌ and ሉ have no common terms, and taken together, their union contains all natural numbers, from 2 onward. In normal cases, both ሌ and ሉ are indexed from the 1 onward. The sequences ሔ and ሑ should satisfy the same condition. Then we call the operation

$$(\text{ሌ}, \text{ሉ}) \searrow (\text{ሔ}, \text{ሑ})$$

the entangling of complementary pair (ሌ, ሉ) with the complementary pair (ሔ, ሑ), defined as an integer sequence a with the following implicit recursive definition:

$$a(1) = 1, \text{ and for all } n > 1,\ a(\text{ሌ}(n)) = \text{ሔ}(a(n)) \text{ and } a(\text{ሉ}(n)) = \text{ሑ}(a(n)).$$

The above says that each n-th term of sequence ሌ should be mapped to the $a(n)$-th term of ሔ, and likewise, each n-th term of sequence ሉ should be mapped to the $a(n)$-th term of ሑ. This forms a well-defined function of natural numbers because the sequences (range of the corresponding functions) ሌ and ሉ are injective on $\mathbb{N}$, thus they have well-defined left inverse functions ሌ^{-1} and ሉ^{-1} and also the indicator functions $1_{\text{ሌ}}$ and $1_{\text{ሉ}}$. With their help, we can rewrite the above definition as an explicit recurrence:

$$a(1) = 1, \text{ and for all } n > 1,\ a(n) = \text{ሔ}(a(\text{ሌ}^{-1}(n))) \text{ when } 1_{\text{ሌ}}(n) = 1$$
$$a(n) = \text{ሑ}(a(\text{ሉ}^{-1}(n))) \text{ when } 1_{\text{ሌ}}(n) = 0.$$

which form can be directly converted to a short recursive program in most programming languages, provided we can find all the required components ሌ^{-1}, ሉ^{-1} and either one of $1_{\text{ሌ}}$ or $1_{\text{ሉ}}$.

For lack of space, the proofs of the following properties are left to the reader. The mathematical induction should suffice for most:

We can swap the order of complementary components inside each pair, if we do it to both pairs at the same time:

$$(\text{ሌ}, \text{ሉ}) \searrow (\text{ሔ}, \text{ሑ}) = (\text{ሉ}, \text{ሌ}) \searrow (\text{ሑ}, \text{ሔ}).$$

If we form an entanglement in opposite order, the result is the inverse of the original and their composition (in either order) is the identity permutation on natural numbers: 1, 2, 3, 4, 5, …, sequence A000027 in OEIS:

$$(\text{ሌ}, \text{ሉ}) \searrow (\text{ሔ}, \text{ሑ}) \circ (\text{ሔ}, \text{ሑ}) \searrow (\text{ሌ}, \text{ሉ}) = (\text{ሔ}, \text{ሑ}) \searrow (\text{ሌ}, \text{ሉ}) \circ (\text{ሌ}, \text{ሉ}) \searrow (\text{ሔ}, \text{ሑ}) = \text{id}.$$

As a corollary of above two rules, the entanglement of any complementary pair (ሌ, ሉ) with the same pair in opposite order (ሉ, ሌ) produces always an involution:

$$[(\text{ሌ}, \text{ሉ}) \searrow (\text{ሉ}, \text{ሌ})]^2 = \text{id}.$$

Useful fact of entanglements is that they obey a "chain rule", allowing us to cancel intermediate components just like we were multiplying rational numbers. Note that for this to work nicely in notation, I have to define a composition of two functions, $f \circ g : X \to Z$, in this order: $(f \circ g)(x) = g(f(x))$. Also, for this axiom I need two more Ethiopic letters, ቤ [be] and ቡ [bu]:

$$(\text{ሌ}, \text{ሉ}) \searrow (\text{ሔ}, \text{ሑ}) \circ (\text{ሔ}, \text{ሒ}) \searrow (\text{ቤ}, \text{ቡ}) = (\text{ሌ}, \text{ሉ}) \searrow (\text{ቤ}, \text{ቡ}).$$

Examples

From this onward, I shall abuse the above notation a little. I shall write (ሌ, ሊ) ⤩ (ሔ, ሒ) even though either one of the sequences at the either side would include as its initial term number one. In that case it is up to the reader to realize that the initial 1 in that sequence should be skipped. So a sequence description like *odd numbers* (or just *odd*) really means odd numbers greater than one, 3, 5, 7, 9, …, and likewise *even numbers* (or just *even*) refers to nonzero even numbers, 2, 4, 6, 8, …. Unless otherwise noted, one is mapped to itself in the resulting permutation, as before.

Apart from odd and even numbers there are other ways to divide natural numbers to two subsets of approximately 50/50 distribution. For example, the *odious* and *evil* numbers (sequences A000069 and A001969) are numbers whose binary representations have odd or even number of 1-digits respectively. Entangling them with odd and even numbers as *(even, odd)* ⤩ *(evil, odious)* yields the famous *binary reflected Gray code* (A003188). However, its plot (left side of figure 2) belongs just to that category of *too regular* fractals of which I have grown tired off.

Somewhat surprisingly, another famous sequence, *Hofstadter-Conway $10000 sequence* A004001 [4], (defined as *a(n) = a(a(n-1))+a(n-a(n-1))* with *a(1) = a(2) = 1*), yields an interesting permutation, which also respects the boundaries of powers of 2. First, two subsequences of natural numbers A088359 and A087686 are picked, based on whether a number occurs in the range of A004001 only once or more than once. Then the entanglement (A087686, A088359) ⤩ *(even, odd)* gives us a permutation A267111, whose plot is shown at the right side of Figure 2. Remarkably, it has a much more flowing character than base-2 related permutations usually have.

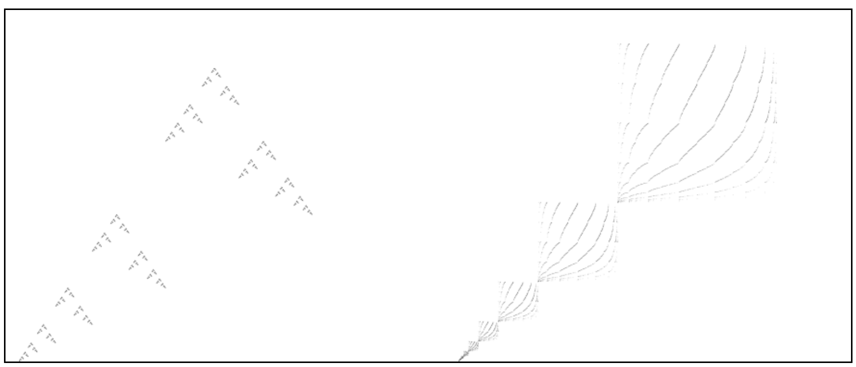

Figure 2: *Two entanglement-permutations with regular base-2 related replicating patterns: On the left, binary reflected Gray code (A003188), and on the right A267111.*

Employing sequences obtained from the number theory (e.g., primes and composites) at the other side of the entanglement operation offers an exit from too much regularity. Indeed, whenever in (ሌ, ሊ) ⤩ (*even*, *odd*) the relative frequency between the components ሌ and ሊ tapers off in more or less continuous fashion, the result will look very much like Matylla's A135141, with the same characteristic "Curtain of Fractal Spray" appearance. As the other subset of the first pair soon becomes "dominant", reducing the resulting permutation modulo $k2^n$ for some k and n clusters many of the values at multiples of 2^n, while the component k of the modulus essentially randomizes the rest. I think this is the reason that even with the simple MIDI-generation script of OEIS [5] with its default pitch modulus 88, some of these sequences generate (*subjectively*) interesting rhythmic patterns. Try A245701 for example.

Variations of the sieve of Eratosthenes include Ulam's *Lucky*/*Unlucky* numbers, and *Ludic*/*Nonludic* numbers. Entangling (*ludic*,*nonludic*)⤰(*primes*,*composites*) produces A255422, with a "comet-like appearance" (see fig 3).

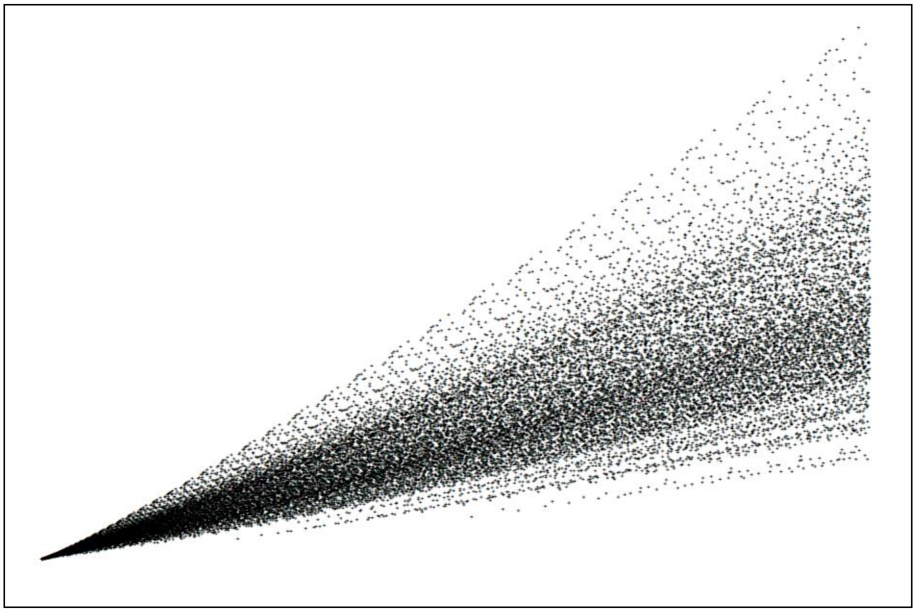

Figure 3: *Scatter plot of A255422: (ludic,nonludic) ⤰ (primes, composites).*

Chebyshev bias [6] is an observation that for the long time, the primes of the form 4*n*+1 are less numerous than those of the form 4*n*+3. The sequences A080147 and A080148 give their indices among all primes. Entangling them with each other yields "Chebyshev's bat", show in Fig. 4.

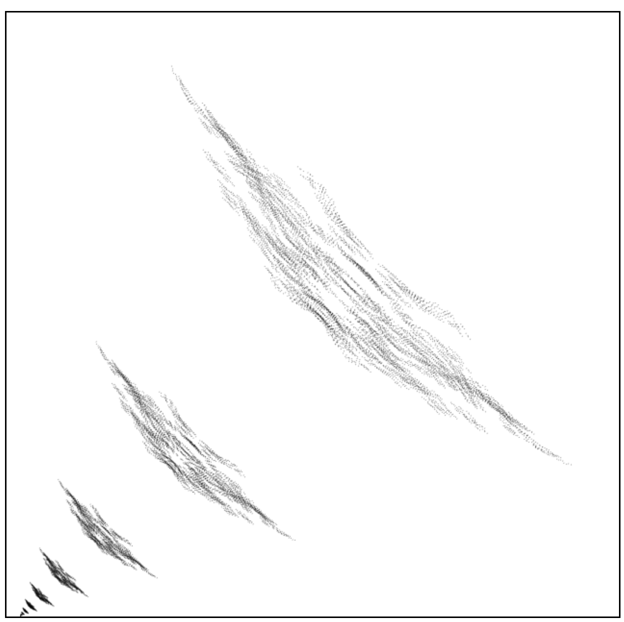

Figure 4: *Scatter plot of "Chebyshev's bat" A267107: (A080147, A080148) ⤰ (A080148, A080147).*

We can approach number sieves in a more piecemeal way than just by using their end results. From sieves like Eratosthenes, Ludic and Lucky, we can from each form a square array, where each row *k* lists all the numbers sieved off at the *k*-th stage of sieve. What is common to all three is that the even numbers (together with 1) are removed in the first batch. For each such array of sieved numbers, we form a *vertical successor function s(n)* that will return the next number immediately below *n* in the same column of that array. By convention, $s(1) = 1$. Because even numbers occur on the first row, they are not in the range of *s*, while each odd number occurs exactly once in the range of such *s*. Thus (*even numbers*, {*s(n) for* n=2..}) is a complementary pair of sequences in the set $\mathbb{N} \setminus \{1\}$, and this pair can be entangled with any another complementary pair. Recall that the definition of entanglement does not require any of the component sequences to be monotonic.

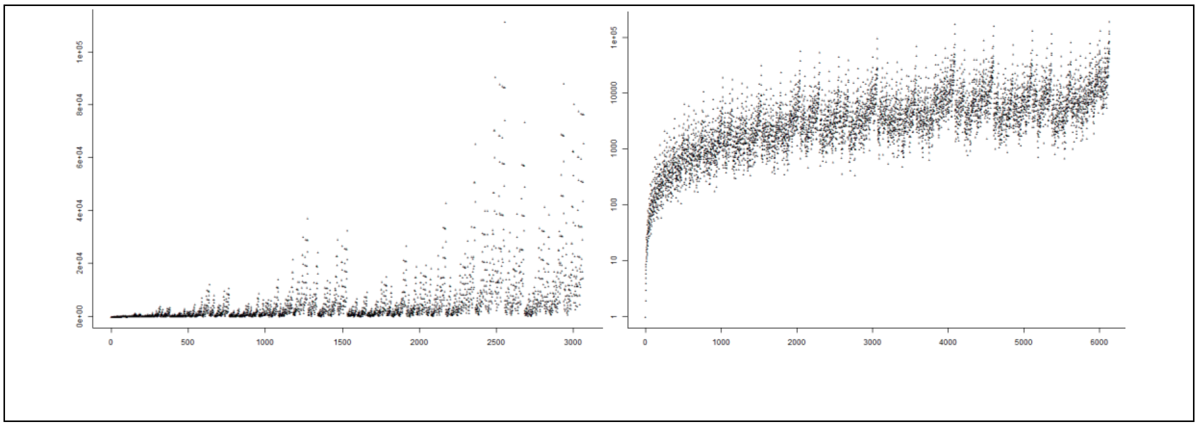

Figure 5: A269865: (even,odd) *) ⤩ (even, A250469). Straight and logarithmic scatter plot.*

If we do the same entanglement slightly differently, we get A252755, entangled as (*even*,*odd*) ⤩$_{\{0\to1,\ 1\to2\}}$ (*even*, A250469). Here the new notation ⤩$_{\{0\to1,\ 1\to2\}}$ means the same as ⤩, except that the resulting permutation maps 0 to 1, 1 to 2, and entangles the rest as before. The resulting scatter plot (fig. 6) looks remarkably different.

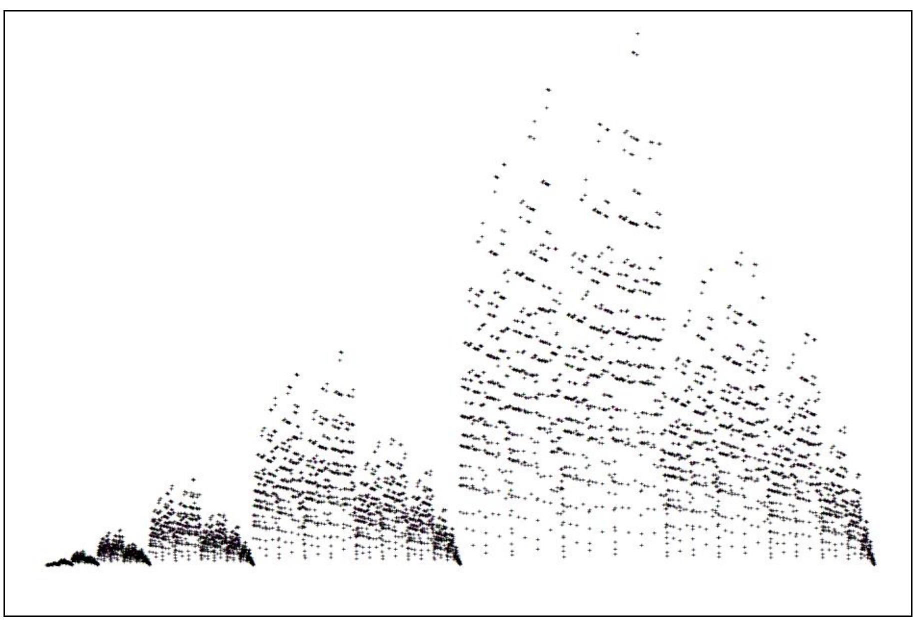

Figure 6: *Scatter plot of A252755: (even,odd)* ⤩$_{\{0\to1,\,1\to2\}}$ *(even, A250469).*

Discussion

I have presented a generic formula for creating permutations from almost any complementary pairs of integer sequences, no matter in which field of discrete mathematics they originate from. For example, A269168 involves one-dimensional cellular automaton, in A269366 the other of complementary pairs is produced by a greedy algorithm, and A269397 entangles “beanstalk-sequences”. It would be interesting to know what actually happens in such entanglements. The most of them clearly exhibit fractal-like scale-invariance (although there are also more chaotic exceptions, e.g. A269863). What portion of the overall behavior is inherited from the sequences to be entangled (e.g. their relative long-term statistical distribution on the other hand, and any unique quirks and starting conditions on the other hand)? What part is just the general effect of the self-amplifying feedback-process inherent in the recursive definition? What is the meaning of “chain rule”? Is it “de-entangling” or “re-entangling” or even “re-encoding”? Sometimes it works even as a flipping of the graph in X/Y-plane, e.g. when composing other entanglement involving odd and even numbers with A054429 = (*even*, *odd*) ⤰ (odd, *even*). Also, entanglements of the form (*even*, *odd*) ⤰ (ሖ, ሑ) give an interesting binomial distribution of “Markovian iterations” of functions ሖ and ሑ, which is especially interesting if the other function is completely regular and the other much more chaotic.

Although the scatter plots of these sequences look pretty, I didn’t mean to stop there. Various prospective avenues in visual arts that immediately come to my mind are L-systems [7] or T-sequences used in weaving [8]. When the magnitude of terms grows quickly very large, one has to decide what part of the information contained in each term should be fed as input to such algorithms (e.g. by modular arithmetic). In musical realm, some of the sequences could probably make more sense as source for raw sound data with further signal processing, while some would work at the note-level, but with much more sophisticated pitch-mappings [9] than “modulo 88”.

References

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