ON INTEGER SEQUENCE A128135

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The purpose of this note is to show how sequence A128135 is obtained by a greedy integer recurrences. To this end, define $a_1 = 1$ and, for $n \geq 2$, let a_n be the least positive integer such that the average of $a_1, \ldots, a_{n-1}$ is a power of 2. We prove the following result.

Theorem 1. *For every positive integer n we have*

$$a_n = \begin{cases} (n+1)2^{\frac{n}{2}-1} & \text{if } n \text{ is even} \\ 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Set $s_n = \sum_{i=1}^{n} a_i$. We claim that $s_n = n2^{\lfloor \frac{n}{2} \rfloor}$ (A132344). We proceed by induction on n. The base case, namely $n = 1$, obviously holds. Now, assume that the assertion holds for every $1 \leq i \leq n-1$, where $n \geq 2$. Let ℓ be a nonnegative integer such that $s_n = n2^{\ell}$. We have

$$a_n = s_n - s_{n-1} = n2^{\ell} - (n-1)2^{\lfloor \frac{n-1}{2} \rfloor}. \tag{1}$$

Clearly, if $\ell \geq \lfloor \frac{n-1}{2} \rfloor$, then $a_n > 0$. We claim that the converse also holds. Indeed, suppose that $a_n > 0$ but $\ell < \lfloor \frac{n-1}{2} \rfloor$. Then

$$n2^{\lfloor \frac{n-1}{2} \rfloor - 1} - (n-1)2^{\lfloor \frac{n-1}{2} \rfloor} > 0 \iff \frac{n}{n-1} > 2.$$

But since $n \geq 2$, we have $\frac{n}{n-1} \leq 2$.

Now, assume that n is even. If $\ell = \lfloor \frac{n-1}{2} \rfloor$, then, by (1) and the induction hypothesis, $a_n = 2^{\lfloor \frac{n-1}{2} \rfloor} = 2^{\frac{n-2}{2}} = a_{n-1}$, in violation of the distinctness condition. Trying the next best candidate $\ell = \lfloor \frac{n-1}{2} \rfloor + 1$, we have, by (1),

$$a_n = n2^{\lfloor \frac{n-1}{2} \rfloor + 1} - (n-1)2^{\lfloor \frac{n-1}{2} \rfloor} = (n+1)2^{\lfloor \frac{n-1}{2} \rfloor},$$

which is obviously adequate. Thus, $s_n = n2^{\lfloor \frac{n}{2} \rfloor}$.

Consider a_{n+1} now and let ℓ be a nonnegative integer such that $s_{n+1} = (n+1)2^{\ell}$. We have

$$a_{n+1} = s_{n+1} - s_n = (n+1)2^{\ell} - n2^{\lfloor \frac{n}{2} \rfloor}.$$

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Here, $\ell = \lfloor \frac{n}{2} \rfloor$ is possible, leading to $a_{n+1} = 2^{\lfloor \frac{n}{2} \rfloor}$ and $s_{n+1} = (n+1)2^{\lfloor \frac{n}{2} \rfloor}$, concluding the proof of the induction step.

$\square$

References

[1] J. Shallit, Proving properties of some greedily-defined integer recurrences via automata theory, *Theoretical Computer Science* **988** (2024).

[2] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.