

# Deformations of the Hadamard product of power series

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## 1 Introduction

In these notes we introduce novel multiplication operations in the algebra of formal power series  $\mathbb{C}[[x]]$ . A multiplication operation is obtained by deforming the ordinary Hadamard product of power series by the action of an invertible lower triangular matrix  $M$ : we refer to this operation as the  $M$ -Hadamard product. The  $M$ -Hadamard products are commutative and associative  $\mathbb{C}$ -bilinear operators on power series. In Section 3 we consider a particular example of an  $M$ -Hadamard product where the deforming matrix  $M$  is chosen to be Pascal's triangle of binomial coefficients  $P = \left(\binom{n}{k}\right)$ . An explicit formula is found for the  $P$ -Hadamard product of monomial polynomials. Dukes and White [DuWh'16], in their study of the combinatorics of web diagrams and web matrices, introduced a commutative and associative  $\mathbb{C}$ -bilinear product of power series, which they named the black diamond product. In Section 4 we show the black diamond product is the same as our  $P$ -Hadamard product. Dukes and White gave examples of polynomial sequences of combinatorial interest, such as the Fubini polynomials and the shifted Legendre polynomials, which have simple expressions in terms of the black diamond product. We give several further examples of this type. We conclude in Section 5 by briefly mentioning other  $M$ -Hadamard products that have some combinatorial interest.

## 2 Deforming the Hadamard product of power series

### 2.1 The Hadamard product

We recall the definition of the Hadamard product of power series.

DEFINITION

**D1:** The Hadamard product  $A(x) * B(x)$  of the power series

$A(x) = \sum_{n=0}^{\infty} a(n)x^n \in \mathbb{C}[[x]]$  and  $B(x) = \sum_{n=0}^{\infty} b(n)x^n \in \mathbb{C}[[x]]$  is defined to be the power series

$$A(x) * B(x) = \sum_{n=0}^{\infty} a(n)b(n)x^n.$$

FACTS

**F1:** The Hadamard product of power series is clearly commutative and associative and distributes over addition of power series.

**F2:** The multiplicative identity element for the algebra  $\mathbb{C}[[x]]$  equipped with the Hadamard product is the power series  $1 + x + x^2 + \dots = \frac{1}{1-x}$ .

**F3:** The set of monomial polynomials  $\{x^n\}_{n \geq 0}$  forms a complete set of mutually orthogonal idempotents in the algebra of power series equipped with the Hadamard product, that is

$$x^i * x^j = \delta_{ij} x^i \quad i, j \geq 0 \quad \text{and} \quad \sum_{i=0}^{\infty} x^i = \text{multiplicative identity.}$$

## 2.2 Deformed Hadamard products

In what follows it will be convenient for us to represent a sequence  $a(n)$  by an infinite column vector. There is an obvious bijective correspondence  $\phi$  between formal power series and their coefficient sequences:

$$A(x) = a(0) + a(1)x + a(2)x^2 + \dots \quad \xleftrightarrow{\phi} \quad \begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \end{pmatrix}.$$

Let  $M$  be a lower triangular matrix. We let  $M$  act on the column vector of coefficients of a power series by matrix multiplication. We can then use the bijection  $\phi$  to pull back this action to an action of  $M$  on the corresponding power series.

### DEFINITIONS

**D2:** We define the action of the lower triangular matrix  $M$  on the power series  $A(x) = \sum_{n \geq 0} a(n)x^n$  by

$$MA(x) = \phi^{-1} \left( M \begin{pmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \end{pmatrix} \right).$$

**D3:** Let  $M$  be an **invertible** infinite lower triangular matrix. Let  $A(x) = \sum_{n \geq 0} a(n)x^n$  and  $B(x) = \sum_{n \geq 0} b(n)x^n$  be formal power series. The

$M$ -Hadamard product of  $A(x)$  and  $B(x)$ , denoted by  $A(x) *_{M} B(x)$ , is the power series defined by

$$A(x) *_{M} B(x) = M^{-1} \left( MA(x) * MB(x) \right). \quad (1)$$

#### FACTS

**F4:** If  $M$  is the identity matrix then the  $M$ -Hadamard product is just the ordinary Hadamard multiplication of power series. We can thus view the  $M$ -Hadamard product as being a deformation of the ordinary Hadamard product by the action of the matrix  $M$ .

**F5:** The power series  $Mx^n$  is the ordinary generating function for the  $n$ th column of the matrix  $M$ .

**F6:** The  $M$ -Hadamard product of power series satisfies the commutative and associative properties and distributes over addition of power series.

**F7:** If  $A(x) = \sum_{i \geq 0} a(i)x^i$  and  $B(x) = \sum_{j \geq 0} b(j)x^j$  then

$$A(x) *_{M} B(x) = \sum_{i,j \geq 0} a(i)b(j) \left( x^i *_{M} x^j \right).$$

Thus knowledge of the products  $x^i *_{M} x^j, i, j \geq 0$  suffices to determine the  $M$ -Hadamard product of two power series.

**F8:** It follows easily from fact F3 that the power series  $E_i(x) := M^{-1}x^i$ , for  $i = 0, 1, 2, \dots$  form a complete set of orthogonal idempotents in the algebra of power series  $\mathbb{C}[[x]]$  equipped with the  $M$ -Hadamard product; that is

$$E_i *_{M} E_j = \delta_{ij} E_i \quad i, j \geq 0$$

and

$$\sum_{i=0}^{\infty} E_i = M^{-1} \frac{1}{1-x} = \text{multiplicative identity.}$$

Every power series  $A(x)$  has an idempotent expansion  $A(x) = \sum_{n=0}^{\infty} a(n)E_n(x)$ , where the coefficients  $a(n)$  are determined by the power series expansion  $MA(x) = \sum_{n=0}^{\infty} a(n)x^n$ .

**F9:** If  $A(x) = \sum_{n=0}^{\infty} a(n)E_n(x)$ ,  $B(x) = \sum_{n=0}^{\infty} b(n)E_n(x)$  are the expansions of the powers series  $A(x)$  and  $B(x)$  in terms of the basis of orthogonal idempotents  $E_n(x)$  then

$$A(x) *_{M} B(x) = \sum_{n=0}^{\infty} a(n)b(n)E_n(x).$$

It follows inductively that the  $k$ -fold product

$$\underbrace{A(x) *_{M} A(x) *_{M} \cdots *_{M} A(x)}_{k \text{ factors}} = \sum_{n=0}^{\infty} a(n)^k E_n(x).$$

In the next section we study a particular example of the  $M$ -Hadamard product.

### 3 An example of a deformed Hadamard product

Let  $P$  denote Pascal's triangle of binomial coefficients  $\binom{n}{k}$ . In this section we work in the algebra of formal power series  $\mathbb{C}[[x]]$  with multiplication of series given by the  $P$ -Hadamard product.

The action of Pascal's triangle  $P$  on a power series  $A(x)$  is known as the binomial transform and takes the form

$$PA(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right), \quad P^{-1}A(x) = \frac{1}{1+x} A\left(\frac{x}{1+x}\right). \quad (2)$$

Hence from (2) and definition D3, the  $P$ -Hadamard product of power series has the following description.

Let  $A(x), B(x) \in \mathbb{C}[[x]]$ . Define sequences  $\alpha(n)$  and  $\beta(n)$  by

$$\frac{1}{1-x} A\left(\frac{x}{1-x}\right) = \sum_{n=0}^{\infty} \alpha(n)x^n, \quad \frac{1}{1+x} B\left(\frac{x}{1+x}\right) = \sum_{n=0}^{\infty} \beta(n)x^n.$$

Then the  $P$ -Hadamard product  $C(x) := A(x) \underset{P}{*} B(x)$  is determined by the equation

$$\frac{1}{1-x} C\left(\frac{x}{1-x}\right) = \sum_{n=0}^{\infty} \alpha(n)\beta(n)x^n. \quad (3)$$

A result of E. Borel [Stan'02, p. 207] states that the Hadamard product of a pair of rational power series is again a rational power series. In addition, Stanley [Stan'80, Theorem 2.10] proves the Hadamard product of a pair of D-finite power series is again a D-finite power series. It is easy to see that both the binomial transform and its inverse (2) send rational power series to rational power series and D-finite power series to D-finite power series (see [Stan'80, Example 2.9]). Combining these observations with (3) leads to the following result.

**Proposition 1.** *If  $A(x), B(x) \in \mathbb{C}[[x]]$  is a pair of rational power series (resp. a pair of D-finite power series) then the  $P$ -Hadamard product  $A \underset{P}{*} B$  is again a rational power series (resp. D-finite power series).  $\square$*

### 3.1 Idempotents

By Fact F5, the orthogonal idempotent power series  $E_i(x) = P^{-1}x^i$  is the generating function of the  $i$ -th column vector of the Riordan array  $P^{-1} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right)$ . Either by this observation or a simple calculation we find

$$\begin{aligned} E_i(x) &= P^{-1}x^i \\ &= \frac{x^i}{(1+x)^{i+1}}. \end{aligned} \quad (4)$$

Applying fact F9 in this case yields the following alternative characterisation of the  $P$ -Hadamard product of power series. If

$$A(x) = \sum_{n=0}^{\infty} a(n) \frac{x^n}{(1+x)^{n+1}}, \quad B(x) = \sum_{n=0}^{\infty} b(n) \frac{x^n}{(1+x)^{n+1}}$$

are the expansions of power series  $A(x)$  and  $B(x)$  in terms of the basis of idempotent power series, then the  $P$ -Hadamard product of  $A(x)$  and  $B(x)$  has the idempotent expansion

$$A(x) \underset{P}{*} B(x) = \sum_{n=0}^{\infty} a(n)b(n) \frac{x^n}{(1+x)^{n+1}}. \quad (5)$$

## EXAMPLES

**E1:** The multiplicative identity element is the constant power series 1 with the idempotent expansion

$$1 = \frac{1}{1+x} + \frac{x}{(1+x)^2} + \frac{x^2}{(1+x)^3} + \cdots$$

**E2:** We also note the easily proved expansion

$$x = \frac{x}{(1+x)^2} + \frac{2x^2}{(1+x)^3} + \frac{3x^3}{(1+x)^4} + \cdots$$

from which we obtain for constants  $r$  and  $s$  the expansion

$$r + sx = \frac{r}{1+x} + \frac{(r+s)x}{(1+x)^2} + \frac{(r+2s)x^2}{(1+x)^3} + \frac{(r+3s)x^3}{(1+x)^4} + \cdots \quad (6)$$

**E3:** It follows inductively from (5) and (6) that the idempotent expansion of the  $n$ -fold product

$$\begin{aligned} (r + sx) \underset{P}{*} \cdots \underset{P}{*} (r + sx) &= \frac{r^n}{1+x} + \frac{(r+s)^n x}{(1+x)^2} + \frac{(r+2s)^n x^2}{(1+x)^3} \\ &+ \frac{(r+3s)^n x^3}{(1+x)^4} + \cdots \end{aligned} \quad (7)$$

### 3.2 A formula for the the $P$ -Hadamard product of monomial polynomials.

By fact F7, to determine the  $P$ -Hadamard product of two power series it is sufficient to know the  $P$ -Hadamard product of monomial polynomials.

**Proposition 2.** *The  $P$ -Hadamard product of two monomial polynomials  $x^m$  and  $x^n$  is given by*

$$x^m \underset{P}{*} x^n = \sum_{k=0}^m \binom{n+k}{k} \binom{n}{m-k} x^{n+k}.$$

**Proof.** The action of Pascal's triangle  $P$  and its inverse  $P^{-1}$  on a monomial polynomial is given by the binomial transform and the inverse binomial transform:

$$Px^j = \frac{x^j}{(1-x)^{j+1}} = \sum_{i \geq 0} \binom{i}{j} x^i, \quad P^{-1}x^j = \frac{x^j}{(1+x)^{j+1}} = \sum_{i \geq 0} (-1)^{i-j} \binom{i}{j} x^i.$$

Therefore, by the definition D3 of the deformed Hadamard product we have

$$\begin{aligned}
x^m * x^n &= P^{-1} (Px^m * Px^n) \\
&= P^{-1} \left( \sum_{i \geq 0} \binom{i}{m} \binom{i}{n} x^i \right) \\
&= \sum_{i \geq 0} \binom{i}{m} \binom{i}{n} P^{-1} x^i \\
&= \sum_{i \geq 0} \binom{i}{m} \binom{i}{n} \sum_{N \geq 0} (-1)^{N-i} \binom{N}{i} x^N \\
&= \sum_{N \geq 0} \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} \binom{i}{m} \binom{i}{n} x^N \\
&= \sum_{N \geq 0} s(N) x^N, \tag{8}
\end{aligned}$$

where we define

$$s(N) = \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} \binom{i}{m} \binom{i}{n},$$

a sum dependent on the parameters  $m$  and  $n$ . Clearly,  $s(N) = 0$  for  $N < n$  and  $s(n) = \binom{n}{m}$ . We find a closed form for this sum using Maple's sumtools package. The Maple code

```

with(sumtools):
sumrecursion((-1)^{N-i} \binom{N}{i} \binom{i}{m} \binom{i}{n}, i, s(n));

```

returns the recurrence equation

$$s(N)(N-n)(N-m) = N(m+n+1-N)s(N-1).$$

It is easy to check that the product of binomials  $p(N) := \binom{N}{N-n} \binom{n}{m+n-N}$  satisfies the same recurrence equation. Also  $p(N)$  is zero for  $N < n$  with  $p(n) = \binom{n}{m} = s(n)$ . We conclude that  $p(N) = s(N)$  for all  $N$ . Hence (8) becomes

$$x^m * x^n = \sum_N \binom{N}{N-n} \binom{n}{m+n-N} x^N. \tag{9}$$

The coefficient of  $x^N$  in the series on the right-hand side of (9) is zero if  $N$  lies outside the closed interval  $[n, m + n]$ . Therefore, if we write  $N = n + k$ , we can recast (9) in the form

$$x^m \underset{P}{*} x^n = \sum_{k=0}^m \binom{n+k}{k} \binom{n}{m-k} x^{n+k}$$

completing the proof of the proposition.  $\square$

## 4 The black diamond product

Web diagrams are generalisations of Feynman diagrams used in the calculation of scattering amplitudes in non-abelian quantum field theory. Dukes and White [DuWh'16], in their study of the combinatorics of web diagrams, introduced a novel multiplication of power series, which they named the black diamond product and denoted by the symbol  $\blacklozenge$ . The  $\blacklozenge$  operator is a  $\mathbb{C}$ -bilinear operator on power series. By [DuWh'16, Definition 8 and Example 10], the black diamond product of power series  $A(x) = \sum_{i \geq 0} a_i x^i$  and

$B(x) = \sum_{i \geq 0} b_i x^i$  is given by

$$A(x) \blacklozenge B(x) = \sum_{k \geq 0} x^k \sum_{i_1, i_2 \geq 0} a_{i_1} b_{i_2} \left( \left( \binom{k}{i_1, i_2} \right) \right)^\star$$

where  $\left( \left( \binom{k}{i_1, i_2} \right) \right)^\star = \binom{k}{k-i_1, k-i_2, i_1+i_2-k}$ .

Applying this formula to the particular case  $A(x) = x^m, B(x) = x^n$  yields

$$x^m \blacklozenge x^n = \sum_k \binom{k}{k-n, k-m, n+m-k} x^k$$

Only summands with  $k \geq \max(n, m)$  contribute to the sum. On replacing the dummy variable  $k$  with  $n+k$  we obtain

$$\begin{aligned} x^m \blacklozenge x^n &= \sum_k \binom{n+k}{k, n+k-m, m-k} x^{n+k} \\ &= \sum_{k=0}^m \binom{n+k}{k} \binom{n}{m-k} x^{n+k}. \end{aligned}$$

Comparing this result with Proposition 2 we see that Dukes and White's black diamond product  $\blacklozenge$  is the same operation as our  $P$ -Hadamard product. We record this fact in the following result.



**Proposition 3.** *Let  $A(x)$  and  $B(x)$  be power series. Then*

$$A(x) \underset{P}{*} B(x) = A(x) \blacklozenge B(x). \square$$

For the remainder of these notes we shall use the symbol  $\blacklozenge$  to denote the  $P$ -Hadamard product.

#### EXAMPLES

**E4:** 
$$x \blacklozenge x^n = nx^n + (n+1)x^{n+1}$$

**E5:** 
$$x^2 \blacklozenge x^n = \binom{n}{2}x^n + 2\binom{n+1}{2}x^{n+1} + \binom{n+2}{2}x^{n+2}$$

**E6:** 
$$x^3 \blacklozenge x^n = \binom{n}{3}x^n + 3\binom{n+1}{3}x^{n+1} + 3\binom{n+2}{3}x^{n+2} + \binom{n+3}{3}x^{n+3}$$

**E7:** [DuWh'16, p.10]  $x^n \blacklozenge x^n = \sum \binom{n+k}{k} \binom{n}{k} x^{n+k} = x^n P_n(2x+1)$ , where  $P_n(2x+1) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k$  is a shifted Legendre polynomial, the  $n$ th row polynomial of [A063007](#). In Section 4.4 below we give several other examples relating the shifted Legendre polynomials to black diamond products.

#### NOTATION

**N1:** We abbreviate the  $n$ -fold product  $A(x) \blacklozenge \cdots \blacklozenge A(x)$  to  $A(x) \blacklozenge^n$  with the convention that  $A(x) \blacklozenge^0 = 1$ .

In the following sections we give examples of polynomials of combinatorial interest that have simple expressions in terms of the black diamond product.

### 4.1 The black diamond product and Stirling numbers

#### EXAMPLES

**E8:** [DuWh'16, eqn. 4] The  $n$ -fold product

$$x \blacklozenge^n = \sum_{k=0}^n k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k \quad n = 0, 1, 2, \dots, \quad (10)$$

where  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  denotes the Stirling numbers of the second kind. The polynomials on the right-hand side of (10) are called the Fubini polynomials and denoted  $\mathcal{F}_n(x)$ . They are also known as the ordered Bell polynomials or geometric polynomials. The Fubini polynomials are the row generating polynomials of A019538 and also A131689. (10) is easily proved by induction, making use of the result  $x \blacklozenge x^n = nx^n + (n+1)x^{n+1}$  (Example E4) and the recurrence equation

$$T(n, k) = kT(n-1, k) + kT(n-1, k-1)$$

satisfied by the numbers  $T(n, k) := k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ .

Setting  $r = 0, s = 1$  in Example E3 yields an expansion for the Fubini polynomial  $\mathcal{F}_n(x)$  for  $n \geq 1$ :

$$\begin{aligned} \mathcal{F}_n(x) &= x \blacklozenge^n \\ &= \frac{x}{(1+x)^2} + \frac{2^n x^2}{(1+x)^3} + \frac{3^n x^3}{(1+x)^4} + \cdots \end{aligned} \quad (11)$$

This expansion is due to Tanny [Tann'75].

**E9:** There is a dual relation to Example E8 involving the Stirling cycle numbers  $\left[ \begin{matrix} n \\ k \end{matrix} \right]$  (see A130534):

$$\begin{aligned} n!x^n &= x \blacklozenge (x-1) \blacklozenge (x-2) \blacklozenge \cdots \blacklozenge (x-n+1). \\ &= \sum_{k=0}^n (-1)^{n-k} \left[ \begin{matrix} n \\ k \end{matrix} \right] x \blacklozenge^k. \end{aligned} \quad (12)$$

A simple inductive proof of this identity can be given using Example E4 and the well-known recurrence for the Stirling cycle numbers

$$\left[ \begin{matrix} n+1 \\ k \end{matrix} \right] = \left[ \begin{matrix} n \\ k-1 \end{matrix} \right] + n \left[ \begin{matrix} n \\ k \end{matrix} \right].$$

**E10:** A shifted version of Example E9 is

$$\begin{aligned} n!(1+x)^n &= (x+1) \blacklozenge (x+2) \blacklozenge \cdots \blacklozenge (x+n) \\ &= \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] x \blacklozenge^k. \end{aligned} \quad (13)$$

These are the row polynomials of A196347.

**E11:** We also note the following shifted versions of Example E8, again easily proved by induction:

$$(1+x)^{\blacklozenge n} = \sum_{k=0}^n k! \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} x^k. \quad (14)$$

These are the row polynomials of A028246 (but with a factor of  $x$  removed).

Setting  $r = 1, s = 1$  in Example E3 yields the expansion

$$(1+x)^{\blacklozenge n} = \frac{1^n}{1+x} + \frac{2^n x}{(1+x)^2} + \frac{3^n x^2}{(1+x)^3} + \frac{4^n x^3}{(1+x)^4} + \cdots. \quad (15)$$

Comparison with the expansion (11) yields

$$(1+x)^{\blacklozenge n} = \frac{(1+x)}{x} \mathcal{F}_n(x), \quad n \geq 1. \quad (16)$$

Expanding the left-hand side of the above by the binomial theorem and using (10) produces an identity satisfied by the Fubini polynomials [MiTa'17]:

$$x \sum_{k=0}^n \binom{n}{k} \mathcal{F}_k(x) = (1+x) \mathcal{F}_n(x), \quad n \geq 1. \quad (17)$$

**E12:** The row polynomials of A038719 are given by

$$(2+x)^{\blacklozenge n} = \sum_{k=0}^n k! \left\{ \begin{matrix} n+2 \\ k+2 \end{matrix} \right\}_2 x^k, \quad (18)$$

where  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_2$  denotes the 2-Stirling number of the second kind. See array [A143494](#).

Setting  $r = 2, s = 1$  in Example E3 yields the expansion

$$(2+x)^{\blacklozenge n} = \frac{2^n}{1+x} + \frac{3^n x}{(1+x)^2} + \frac{4^n x^2}{(1+x)^3} + \frac{5^n x^3}{(1+x)^4} + \cdots. \quad (19)$$

**E13:** The polynomials  $(1+2x)^{\blacklozenge n}$  are the row polynomials of A145901, the triangle of  $f$ -vectors of the simplicial complexes dual to the permutohedra of type  $B_n$ .

Setting  $r = 1, s = 2$  in Example E3 yields the expansion

$$(1 + 2x)^{\blacklozenge n} = \frac{1^n}{1+x} + \frac{3^n x}{(1+x)^2} + \frac{5^n x^2}{(1+x)^3} + \frac{7^n x^3}{(1+x)^4} + \cdots \quad (20)$$

We also note the binomial expansion

$$\begin{aligned} (1 + 2x)^{\blacklozenge n} &= \sum_{k=0}^n \binom{n}{k} 2^k x^{\blacklozenge k} \\ &= \sum_{k=0}^n \binom{n}{k} 2^k \mathcal{F}_k(x) \end{aligned}$$

expressing the row polynomials of [A145901](#) in terms of the Fubini polynomials.

**E14:** The polynomials  $(1 + 3x)^{\blacklozenge n}$  are the row polynomials of [A284861](#).

Setting  $r = 1, s = 3$  in Example E3 yields the expansion

$$(1 + 3x)^{\blacklozenge n} = \frac{1^n}{1+x} + \frac{4^n x}{(1+x)^2} + \frac{7^n x^2}{(1+x)^3} + \frac{10^n x^3}{(1+x)^4} + \cdots \quad (21)$$

The binomial expansion

$$\begin{aligned} (1 + 3x)^{\blacklozenge n} &= \sum_{k=0}^n \binom{n}{k} 3^k x^{\blacklozenge k} \\ &= \sum_{k=0}^n \binom{n}{k} 3^k \mathcal{F}_k(x) \end{aligned}$$

expresses the row polynomials of [A284861](#) in terms of the Fubini polynomials.

## 4.2 The black diamond product of a geometric series and a power series

**Proposition 4.** Let  $A(x) \in \mathbb{C}[[x]]$  and let  $\alpha, \beta$  be a pair of complex numbers. Then

$$\frac{1}{1 - \alpha x} \blacklozenge A(\beta x) = \frac{1}{1 - \alpha x} A\left(\frac{(1 + \alpha)\beta x}{1 - \alpha x}\right). \quad (22)$$

**Proof.**

Recall the action of the binomial transform on a power series

$$F(x) = \sum_{n \geq 0} f(n)x^n:$$

$$P(F(x)) = \frac{1}{1-x} F\left(\frac{x}{1-x}\right) = \sum_{n \geq 0} \left( \sum_{k=0}^n \binom{n}{k} f(k) \right) x^n. \quad (23)$$

Let  $C(x)$  denote the power series on the right-hand side of (22):

$$C(x) = \frac{1}{1-\alpha x} A\left(\frac{(1+\alpha)\beta x}{1-\alpha x}\right). \quad (24)$$

By definition D3, the proposition is equivalent to the identity

$$PC(x) = P\left(\frac{1}{1-\alpha x}\right) * P(A(\beta x)). \quad (25)$$

We verify this is true by comparing the coefficients of  $x^n$  on both sides of (26).

Firstly, we find the power series expansion of the left-hand side of (25). Let  $A(x) = \sum_{n \geq 0} a(n)x^n$ . Then by (23)

$$\begin{aligned} P(C(x)) &= \frac{1}{1-x} C\left(\frac{x}{1-x}\right) \\ &= \frac{1}{(1-(1+\alpha)x)} A\left(\beta \frac{(1+\alpha)x}{1-(1+\alpha)x}\right) \quad \text{by (24)} \\ &= \frac{1}{1-X} A\left(\frac{\beta X}{1-X}\right) \quad \text{where } X = (1+\alpha)x \\ &= \sum_{n \geq 0} \left( \sum_{k=0}^n \binom{n}{k} a(k)\beta^k \right) X^n \quad \text{by (23)} \\ &= \sum_{n \geq 0} \left( \sum_{k=0}^n \binom{n}{k} a(k)\beta^k \right) (1+\alpha)^n x^n \end{aligned} \quad (26)$$

On the other hand we have

$$\begin{aligned} P\left(\frac{1}{1-\alpha x}\right) &= \frac{1}{1-(1+\alpha)x} \\ &= \sum_{n \geq 0} (1+\alpha)^n x^n \end{aligned}$$

and by (23)

$$P(A(\beta x)) = \sum_{n \geq 0} \left( \sum_k \binom{n}{k} a(k)\beta^k \right) x^n.$$

Hence the Hadamard product

$$\begin{aligned} P\left(\frac{1}{1-\alpha x}\right) * P(A(\beta x)) &= \sum_{n \geq 0} \left( \sum_k \binom{n}{k} a(k)\beta^k \right) (1+\alpha)^n x^n. \\ &= P(C(x)) \end{aligned}$$

by (26), thus proving (25) and completing the proof of the proposition.  $\square$

## EXAMPLES

**E15:** Take  $A(x) = \frac{1}{1-x}$  in Proposition 4. Then (22) becomes

$$\frac{1}{1-\alpha x} \blacklozenge \frac{1}{1-\beta x} = \frac{1}{1-(\alpha+\beta+\alpha\beta)x}. \quad (27)$$

In particular,

$$\frac{1}{1-x} \blacklozenge \frac{1}{1-\frac{xy}{2}} = 1 + (1+y)x + (1+y)^2x^2 + (1+y)^3x^3 + \dots \quad (28)$$

is the ordinary generating function for Pascal's triangle expressed as the black diamond product of two geometric series.

**E16:** We can generate further identities by differentiating (27) with respect to either  $\alpha$  or  $\beta$ : for example, if we differentiate (27) with respect to  $\alpha$  and then with respect to  $\beta$  we obtain

$$\frac{x}{(1-\alpha x)^2} \blacklozenge \frac{x}{(1-\beta x)^2} = \frac{x(1+x(\alpha+\beta+\alpha\beta+2))}{(1-(\alpha+\beta+\alpha\beta)x)^3}. \quad (29)$$

Setting  $\alpha = \beta = -1$  in (29) confirms that  $E_2(x) = x/(1+x)^2$  is an idempotent in the the black diamond algebra.

Setting  $\alpha = \beta = -2$  in (29) gives

$$\frac{x}{(1+2x)^2} \blacklozenge \frac{x}{(1+2x)^2} = x(1+2x).$$

Here we have an example of the black diamond product of a pair of rational functions resulting in a polynomial. For a generalisation of this identity see exercise Ex2 below.

### 4.3 The black diamond product and Laguerre polynomials

The ordinary generating function for the Laguerre polynomials  $L_n(y)$  takes the form

$$\sum_{n \geq 0} L_n(y)x^n = \frac{1}{1-x} \exp\left(\frac{-xy}{1-x}\right).$$

Applying Proposition 4 with  $A(x) = \exp(x)$ ,  $\alpha = 1$  and  $\beta = -\frac{y}{2}$  we obtain the generating function of the Laguerre polynomials in terms of the black diamond product:

$$\frac{1}{1-x} \blacklozenge e^{-\frac{xy}{2}} = \sum_{n \geq 0} L_n(y) x^n, \quad (30)$$

which should be compared with (2).

More generally, if we take  $A(x) = \exp(x)$ ,  $\alpha = 1$  and  $\beta = -y$  in Proposition 4 we find

$$\frac{1}{1-x} \blacklozenge e^{-xy} = \frac{1}{1-x} \exp\left(\frac{-2xy}{1-x}\right). \quad (31)$$

If we differentiate (31)  $k$  times with respect to  $y$  and then replace  $y$  with  $y/2$  we obtain a generating function for the associated Laguerre polynomials  $L_n^{(k)}(y)$  [ref] in terms of the black diamond product:

$$\begin{aligned} \frac{x^k}{1-x} \blacklozenge e^{-\frac{xy}{2}} &= 2^k x^k \frac{1}{(1-x)^{k+1}} \exp\left(\frac{-xy}{1-x}\right) \\ &= 2^k x^k \sum_{n \geq 0} L_n^{(k)}(y) x^n. \end{aligned} \quad (32)$$

#### 4.4 The black diamond product and Legendre polynomials

We have already seen in Example E7 the identity

$$x^n \blacklozenge x^n = \sum \binom{n+k}{k} \binom{n}{k} x^{n+k} = x^n P_n(2x+1)$$

expressing the shifted Legendre polynomials  $P_n(2x+1)$  in terms of a black diamond product. The following exercises give further relations of this type. The proofs require the use of a computer algebra system to find recurrences for the various binomial sums that arise.

##### EXERCISES

**Ex1:** Show

$$\frac{1}{(1+x)^{n+1}} \blacklozenge \frac{1}{(1+x)^{n+1}} = \frac{P_n(2x+1)}{(1+x)^{n+1}} \quad n = 0, 1, 2, \dots$$

**Ex2:** Show

$$\frac{x^n}{(1+2x)^{n+1}} \blacklozenge \frac{x^n}{(1+2x)^{n+1}} = x^n P_n(2x+1) \quad n = 0, 1, 2, \dots$$

**Ex3:** Show

$$\sum_{k=0}^n \binom{n}{k} x^k \blacklozenge (1+x)^{n-k} = P_n(2x+1), \quad n = 0, 1, 2, \dots$$

## 4.5 Miscellaneous identities

### EXERCISES

**Ex4:** Show

$$\frac{1-x}{1+x} \blacklozenge \frac{1-x}{1+x} = 1$$

**Ex5:** Show

$$\left(\frac{1-2x}{1+2x}\right)^{\blacklozenge n} = \frac{1+(1+(-3)^n)x}{1+2x}$$

**Ex6:** For  $m \geq 0$  and  $n \in \mathbb{Z}$ , show

$$x^m \blacklozenge (1+x)^n = \binom{m+n}{m} x^m (1+x)^n$$

The following identity is conjectural.

**Ex7:** Let  $a \in \mathbb{C}$ . Show

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (x+a)^k \blacklozenge (x+a)^{n-k} \stackrel{?}{=} \begin{cases} (-1)^m \binom{2m}{m} x^m (1+x)^m & n = 2m \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

## 5 Other $M$ -Hadamard products

Finally, we briefly consider two other  $M$ -Hadamard products related to the black diamond product.

1)  $M = \binom{n}{k}_q$  :

One possibility is to investigate the  $M$ -Hadamard product where the deformation matrix  $M = \binom{n}{k}_q$  is the matrix of  $q$ -binomial coefficients. Denote the resulting multiplication operator on power series by  $\blacklozenge_q$ . We can view this operator as a  $q$ -analogue of the black diamond product and investigate  $q$ -analogues of the results of Section 4. As an example, the  $q$ -analogue of (10) appears to be

$$\underbrace{x \blacklozenge_q \dots \blacklozenge_q x}_{\text{nfactors}} = \sum_{k=0}^n q^{\binom{k}{2}} [k]_q! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q x^k \quad n = 0, 1, 2, \dots,$$

where  $[k]_q!$  is the  $q$ -factorial and  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$  is a  $q$ -Stirling number of the second kind.



2)  $M = \left( \binom{n}{k} k! \right) :$

In [Bala'18] we introduced a multiplication of power series called the white diamond product denoted by the symbol  $\diamond$ . The white diamond product is an  $M$ -Hadamard product where the deformation matrix  $M$  is taken to be the matrix  $\left( \binom{n}{k} k! \right)$ . The set of integer polynomials  $\mathbb{Z}[x]$  is closed under the white diamond product. Many well-known sequences of polynomials, including the Bell polynomials, the Lah polynomials, the Laguerre polynomials and the Bessel polynomials, are shown to have simple expressions in terms of the white diamond product. For example, the  $n$ -fold product  $x \diamond \cdots \diamond x$  equals the  $n$ th Bell polynomial  $B_n(x)$ . The idempotents in the white diamond algebra are the series  $e^{-x} \frac{x^i}{i!}$ . The idempotent expansion of the Bell polynomials in the white diamond algebra turns out to be the well-known Dobinski formula

$$B_n(x) = e^{-x} \sum_k k^n \frac{x^k}{k!}.$$

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