Construction of an integer sequence with "musical properties"

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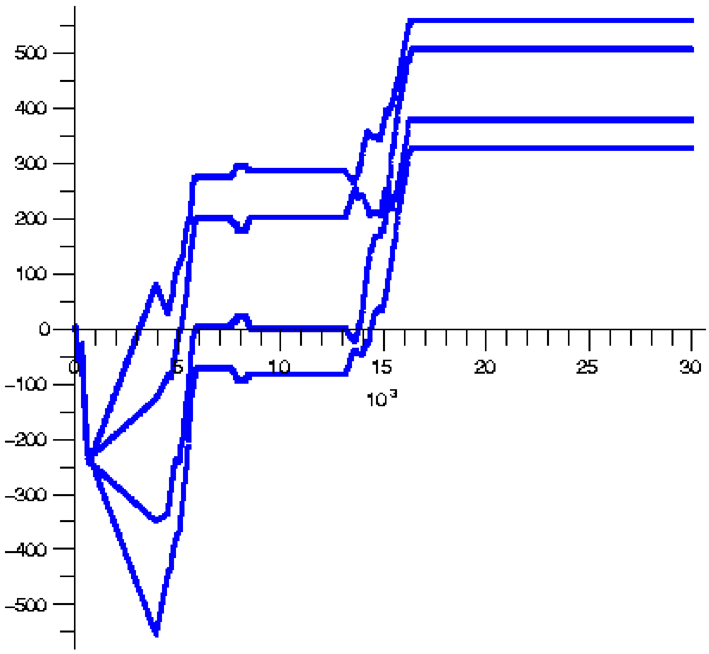
1, 4, 2, 1, 6, 7, 4, 2, 6, 6, 1, 2, 4, 7, 1, 2, 4, -1, -3, 0, 4, -1,-11, -6, -1, -5, -13, -10, -4, -10, -14, -10, -9, -13, -17, -12, -11, -19, -18, -13, -15, -19, -18, -15, -18, -23, -19, -15, -18, -25, -23, -18, -22, -30, -25, -20, -27, -34, -30, -24, -30, -39, -35, -26, -33, -44, -39, -31, -35, -46, -42, -34, -39, -47, -43, -38, -43, -50, -45, -40, -46, -53, -50, -42, -46, -54, -50, -42, -46, -53, -52, -47, -49, -53, -52, -49, -53, -57, -53, -48, -53, -59, -52, -46, -53, -60, -54, -46, -51, -59, -54, -46, -50, -56, -50, -41, -49, -54, -47, -41, -46, -50, -44, -38, -43, -47, -42, -36, -39, -45, -41, -35, -37, -44, -40, -33, -37, -42, -38, -32, -37, -40, -37, -34, -36, -39, -37, -34, -35, -38, -39, -35, -35, -37, -38, -36, -36, -37, -36, -36, -37, -37, -36, -35, -38, -39, -35, -33, -39, -40, -34, -32, -39, -42, -34, -31, -39, -42, -35, -32, -37, -41, -37, -33, -36, -40, -36, -33, -36, -40, -37, -33, -36, -41, -36, -31, -37, -41, -35, -32 ...



Construction of the above sequence

The front page graph plots the first 30000 terms the sequence $(a_n)_{n\in N}$ to be described here.

Let M(4) be the 16 dimensional algebra of all 4 x 4 matrices over the real numbers. Prof. E. Clark noticed that the set of 16 matrices below form a basis for M(4):

$$\bar{i} \simeq \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \bar{j} \simeq \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \bar{k} \simeq \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{i} \simeq \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \underline{j} \simeq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \underline{k} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

along with the following 10 matrices

$$ii, jj, kk, ij, ik, ji, jk, ki, kj, e$$

defined by multiplying the above matrices. For ex.

$$ij = \bar{i} * \underline{j} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Any $x \in M(4)$ can be written as

$$x = x_0\bar{i} + x_1\bar{j} + x_2\bar{k} + x_3\underline{i} + x_4\underline{j} + x_5\underline{k} + x_6 ii + x_7 jj + x_8 kk + x_9 ij + x_{10} ik + x_{11} ji + x_{12} jk + x_{13} ki + x_{14} kj + x_{15} e$$

First Iteration: Calculating the intitial term a_1

Set

$$Y = X = \bar{i} + \frac{1}{4}(ii + jj + kk + e)$$

and

$$Z = \bar{i} - \underline{i} + \frac{1}{2}(jj + kk - jk + kj) + e$$

(Note that $X^{12} = Y^{12} = e$) Let "$A \longrightarrow B$" mean to set (or perhaps better: redefine) A equal B.

Step A (1st Iteration):

$$Y \longrightarrow X \cdot Y = .5(\bar{i} - \underline{i}) + .25(ii + jj + kk) - .75e$$

Step B (1st Iteration):

$$Y \longrightarrow Y + X = (.5(\bar{i} - \underline{i}) + .25(ii + jj + kk) - .75e) + (\bar{i} + .25(ii + jj + kk + e)) = 1.5\bar{i} - .5\underline{i} + .5(ii + jj + kk - e)$$

Let $[X]$ be obtained by setting the coefficients of the basis vectors of X equal to their fractional parts.

Step C (1st Iteration):

$$Z \longrightarrow Z - [Y] = (\bar{i} - \underline{i} + .5(jj + kk - jk + kj) + e) - .5(\bar{i} - \underline{i} + ii + jj + kk - e) = .5(\bar{i} - \underline{i} - ii + kj - jk) + 1.5e$$

Step D (1st Iteration):

$$Y \longrightarrow Y + 2Z = (1.5\bar{i} - .5\underline{i} + .5(ii + jj + kk) - .5e) + (\bar{i} - \underline{i} - ii + kj - jk + 3e)$$

Thus, at the end of the first iteration, we are left with:

$$Y = 2.5\underline{i} - 1.5\bar{i} - .5ii + .5jj + .5kk - jk + kj + 2.5e$$

A computer program returns the output Step **1** (or 1st iteration); see next two pages. The following iterations are obtained by returning to step A and repeating the procedure with the values from the previous iteration. The sequence $(a_n)_{n \in N}$ graphed at the top of the page is obtained by adding the coefficents from the basis vectors $\bar{i}$ and $\underline{i}$. At the end of the first iteration, we get $a_1 = 2.5 - 1.5 = 1$. Had we, instead, added up over other basis vectors, then the sequences so obtained should surely "somehow" be related to $(a_n)_{n \in N}$ (see http://www.crowdog.de/Ausschnitt.html : the sequence $(a_n)_{n \in N}$ is "jes"). Note that the outcome (or "complexity") of a sequence depends on how the element Z is defined. If, for example, we set $Z = X$, then the following (apparently simpler!) sequence results:

1, 3, 4, 3, 1, 6, 9, 1, -2, 7, 7, 2, 1, 5, 5, 4, 3, 5, 5, 5, 4, 4, 6, 5, 2, 4,
7, 5, 2, 5, 8, 5, 3, 5, 7, 6, 4, 4, 7, 6, 4, 5, 7, 6, 5, 6, 5, 5, 5, 7, 6, 4,
6, 7, 4, 4, 7, 7, 4, 3, 6, 9, 5, 1, 5, 10, 6, 0, 5, 10, 6, 1, 5, 10, 7, 2, 3,
9, 7, 3, 4, 8, 8, 3, 2, 8, 9, 3, 2, 7, 8, 5, 3, 5, 7, 6, 4, 4, 7, 6, 4, 5, 7,
6, 5, 6, 5, 5, 5, 7, 6, 4, 6, 7, 4, 4, 7, 7, 4, 3, 6, 9, 5, 1, 5, 10, 6, 0,
5, 10, 6, 1, 5, 10, 7, 2, 3, 9, 7, 3, 4, 8, 8, 3, 2, 8, 9, 3, 2, 7, 8, 5, 3,
5, 7, 6, 4, 4, 7, 6, 4, 5, 7, 6, 5, 6, 5, 5, 5, 7, 6, 4, 6, 7, 4, 4, 7, 7, 4,
3, 6, 9, 5, 1, 5, 10, 6, 0, 5, 10, 6, 1, 5, 10, 7, 2, 3, 9, 7, 3, 4, 8, 8, 3,
2, 8, 9, 3, 2, 7, 8, 5, 3, 5, 7, 6, 4, 4, 7, 6, 4, 5, 7, 6, 5, 6, 5, 5, 5, 7,
6, 4, 6, 7, 4, 4, 7, 7, 4, 3, 6, 9, 5, 1, 5, 10, 6, 0, 5, 10, 6, 1, 5, 10 ...

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**1**  + 2.5'i - 1.5i' - .5'ii' + .5'jj' + .5'kk' - 'jk' + 'kj' + 2.5e

**2**  + 5'i - i' - .5'ii' - 'jk' + 'kj' + .5e

**3**  + 2.5'i - .5i' + .75'ii' + .75'jj' + .75'kk' + .75e

**4**  + 1.5'i - .5i' + 'ii' + 2.5'jj' + 2.5'kk' - 'jk' + 'kj' + 6e

**5**  + 6'i + 1.75'ii' + 3.25'jj' + 3.25'kk' - 2.5'jk' + 2.5'kj' + 7.75e

**6**  + 8'i - i' + 2.75'ii' + 1.25'jj' + 1.25'kk' - 3'jk' + 3'kj' + 5.75e

**7**  + 5.5'i - 1.5i' + .5'ii' - .5'jj' - .5'kk' + .5'jk' - .5'kj' + 1.5e

**8**  + 1.5'i + .5i' - 2.5'ii' + .5'jj' + .5'kk' + 2.5'jk' - 2.5'kj' + 2.5e

**9**  + 2'i + 4i' - .5'ii' + 2.5'jj' + 2.5'kk' + 'jk' - 'kj' + 7.5e

**10**  + 3.5'i + 2.5i' + 5.25'ii' + 3.25'jj' + 3.25'kk' - 1.5'jk' + 1.5'kj' + 7.25e

**11**  + 2.25'i - 1.25i' + 4.5'ii' + 1.5'jj' + 1.5'kk' - 1.25'jk' + 1.25'kj' + 7.5e

**12**  + 1.75'i + .25i' - 1.75'ii' - .75'jj' - .75'kk' + 1.75'jk' - 1.75'kj' + 7.25e

**13**  - .5'i + 4.5i' - 4'ii' - .5'jj' - .5'kk' + 5'jk' - 5'kj' + 4e

**14**  + 1.75'i + 5.25i' - 2'ii' + 1.5'jj' + 1.5'kk' + 3.25'jk' - 3.25'kj' + 4e

**15**  - 2.25'i + 3.25i' - 1.25'ii' - .25'jj' - .25'kk' + 2.25'jk' - 2.25'kj' + 3.75e

**16**  - .5'i + 2.5i' - 4'ii' - 2'jj' - 2'kk' + 3'jk' - 3'kj' + 7e

**17**  + .25'i + 3.75i' - 6.5'ii' - 2'jj' - 2'kk' + 4.25'jk' - 4.25'kj' + 3.5e

**18**  - 6.25'i + 5.25i' - 7.25'ii' - 3.75'jj' - 3.75'kk' + 4.25'jk' - 4.25'kj' + 2.75e

**19**  - 11'i + 8i' - 8'ii' - 6'jj' - 6'kk' + 4'jk' - 4'kj' + 8e
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20 - 9.25'i + 9.25i' - 5.75'ii' - 6.75'jj' - 6.75'kk' + 3.75'jk' - 3.75'kj' + 11.25e

21 - 2'i + 6i' - 4.5'ii' - 6'jj' - 6'kk' + 4.5'jk' - 4.5'kj' + 9.5e

22 - 2.25'i + 1.25i' - 8.5'ii' - 5'jj' - 5'kk' + 6.25'jk' - 6.25'kj' + 1.5e

23 - 13.75'i + 2.75i' - 14.25'ii' - 6.25'jj' - 6.25'kk' + 6.75'jk' - 6.75'kj' + .75e

24 - 17.5'i + 11.5i' - 14'ii' - 9'jj' - 9'kk' + 4'jk' - 4'kj' + 9e

25 - 16.25'i + 15.25i' - 4.5'ii' - 10'jj' - 10'kk' + .75'jk' - .75'kj' + 13.5e

26 - 13'i + 8i' + .75'ii' - 8.75'jj' - 8.75'kk' - 1.5'jk' + 1.5'kj' + 14.75e

27 - 13.75'i + .75i' - 5.75'ii' - 7.25'jj' - 7.25'kk' - 2.75'jk' + 2.75'kj' + 11.25e

28 - 14.75'i + 4.75i' - 14.25'ii' - 8.75'jj' - 8.75'kk' - 3.75'jk' + 3.75'kj' + 8.75

29 - 19'i + 15i' - 12.25'ii' - 12.75'jj' - 12.75'kk' - 4'jk' + 4'kj' + 4.75e

30 - 25.25'i + 15.25i' - 4.5'ii' - 15.5'jj' - 15.5'kk' - 4.25'jk' + 4.25'kj' + 8.5e