Proof that A115431, A116117, and A116135 are the same

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A115421 is "Numbers n such that the concatenation of n with $n-2$ gives a square." A116117 is "Numbers n such that n concatenated with $n-6$ gives the product of two numbers which differ by 4." A116135 is "Numbers n such that n concatenated with $n-3$ gives the product of two numbers which differ by 2."

If $n-2$ has d digits, the concatenation of n with $n-2$ is $(10^d+1)n-2$. This is y^2 if and only if $(10^d+1)n-3 = (y+1)(y-1)$ is the product of two numbers which differ by 2, and if and only if $(10^d+1)n-6 = y^2-6 = (y+2)(y-2)$ is the product of two numbers which differ by 4. The only cases where n can be in one of these sequences and not the others is when $n-2$ and $n-6$ have different numbers of digits: if $n = 10^m + j$ where $2 \le j \le 5$, then $n-2$ has $d = m+1$ digits but $n-6$ has fewer. Thus we must consider when

$$F(j,m) = (10^{m+1}+1)(10^m+j) - 2 = 10^{2m+1} + j \cdot 10^{m+1} + 10^m + j - 2$$

can be a square, for j from 2 to 5. This is in fact the case for $m=0$, $j=5$, as $F(5,0) = 64$ is a square and $10^0+5-3 = 3 > 10^0$ but $10^0+5-6 = 0 < 10^0$, but we still consider 0 to have one digit so this doesn't produce a difference between the sequences. From now on we will assume $m \ge 1$.

Since $F(j,m) \equiv j-2 \mod 10$, $j-2$ must be a square $\mod 10$. This rules out $j=4$ and $j=5$, as 2 and 3 are not squares $\mod 10$. Similarly, $F(j,m) \equiv 2j \mod 9$, so this must be a square $\mod 9$. This rules out $j=3$. The only remaining possibility is $j=2$, with $F(2,m) = 10^{2m+1} + 2 \cdot 10^{m+1} + 10^m = 10^m(10^{m+1}+21)$. Since 10^m and $10^{m+1}+21$ are coprime, for this to be a square would require $10^{m+1}+21$ to be a square, and that is impossible because 21 is not a square $\mod 1000$ (and $F(2,1) = 1210$ is not a square).