Identities for A115110

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Let $E(q)$ denote the Euler product $\prod_{n\geq 1}(1-q^n)$. In [1, Theorem 1], the authors find identities satisfied by the coefficients in the series expansion of $E(q)^r$ for various values of r. In this note we find similar identities satisfied by the coefficients in the series expansion of $E(q)E(-q)$ and state the corresponding result for the coefficients in the series expansion of $(E(q)E(-q))^3$.

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The expansion of $E(q)E(-q)$ begins

$$E(q)E(-q) = 1 - 3q^2 + q^4 + 2q^6 + 2q^8 - q^{10} - 4q^{12} + q^{14} + \cdots .$$

The coefficient sequence $[1, -3, 1, 2, 2, -1, -4, 1, ...]$ is A115110.

Theorem 1. *Let p be a prime of the form $4k+3$. Let $E(q)E(-q) = \sum_{n\geq 0} a(n)q^n$. Then*

(i)

$$a\left(p^2 n + \frac{p^2-1}{12}\right) = \epsilon a(n) \tag{1}$$

where

$$\epsilon = \begin{cases} 1 & p \equiv 7 \text{ or } 23 \pmod{24} \\ -1 & p \equiv 11 \text{ or } 19 \pmod{24} \end{cases} \tag{2}$$

(ii) *If $n > 0$ is coprime to p then*

$$a\left(pn + \frac{p^2-1}{12}\right) = 0. \tag{3}$$

Proof. (i) We recall Euler's series expansion of the Euler product (see, for example, [2]):

$$E(q) = \sum_{m\in\mathbb{Z}} (-1)^m q^{m(3m+1)/2}. \tag{4}$$

We thus have

$$E(q)E(-q) = \sum_{m,n\in\mathbb{Z}} (-1)^{m+n+n(3n+1)/2} q^{m(3m+1)/2+n(3n+1)/2}. \tag{5}$$

The coefficient $a(N)$ of the term q^N in (5) is given by

$$a(N) = \sum_{(m,n)\in S(N)} t(m,n), \tag{6}$$

where

$$t(m,n) = (-1)^{m+n+n(3n+1)/2} \tag{7}$$

and

$$S(N) = \left\{(m,n) \mid \frac{m(3m+1)}{2} + \frac{n(3n+1)}{2} = N, \quad m,n \in \mathbb{Z}\right\}. \tag{8}$$

Since

$$\begin{aligned} \frac{m(3m+1)}{2} + \frac{n(3n+1)}{2} &= N \Leftrightarrow \\ 12m(3m+1) + 12n(3n+1) &= 24N \Leftrightarrow \\ (6m+1)^2 + (6n+1)^2 &= 24N+2, \end{aligned}$$

we can redefine the set $S(N)$ in (8) in the more convenient form

$$S(N) = \left\{(m,n) \mid (6m+1)^2 + (6n+1)^2 = 24N+2, \quad m,n \in \mathbb{Z}\right\}. \tag{9}$$

Notice that as an immediate consequence we have

$$S\left(p^2N + \frac{p^2-1}{12}\right) = \left\{(m,n) \mid (6m+1)^2 + (6n+1)^2 = p^2(24N+2), m,n \in \mathbb{Z}\right\}. \tag{10}$$

We define a mapping on pairs of integers (m,n) by

$$\phi(m,n) = \left(p^*m + \frac{p^*-1}{6},\ p^*n + \frac{p^*-1}{6}\right). \tag{11}$$

where

$$p^* = \begin{cases} p & p \equiv 7 \text{ or } 19 \pmod{24} \\ -p & p \equiv 11 \text{ or } 23 \pmod{24} \end{cases} \tag{12}$$

Notice that p^* is of the form $6k+1$ for some integer k. We shall abuse notation and also use the same symbol ϕ to denote the integer mapping $\phi(m) = p^*m + (p^*-1)/6$.

One easily checks from (7) that

$$t(\phi(m,n)) = \epsilon t(m,n), \tag{13}$$

where $\epsilon = 1$ if $p \equiv 7$ or $23 \pmod{24}$ and $\epsilon = -1$ if $p \equiv 11$ or $19 \pmod{24}$. We claim that ϕ maps the set $S(N)$ bijectively onto the set $S\left(p^2N + \frac{p^2-1}{12}\right)$. The desired result $a\left(p^2N + \frac{p^2-1}{12}\right) = \epsilon a(N)$ will follow from this since by (6) and (13)

$$\begin{aligned}
\epsilon a(N) &= \sum_{(m,n)\in S(N)} \epsilon t(m,n), \\
&= \sum_{(m,n)\in S(N)} t(\phi(m,n)) \\
&= \sum_{\phi(m,n)\in S(p^2N+(p^2-1)/12)} t(\phi(m,n)) \\
&= a\left(p^2N + \frac{p^2-1}{12}\right)
\end{aligned}$$

by (6).

Firstly, let us prove that ϕ maps $S(N)$ into $S\left(p^2N + \frac{p^2-1}{12}\right)$. Suppose $(m,n) \in S(N)$. We have by (11)

$$\begin{aligned}
(6\phi(m)+1)^2 + (6\phi(n)+1)^2 &= (6p^*m + p^*)^2 + (6p^*n + p^*)^2 \\
&= p^2\left\{(6m+1)^2 + (6n+1)^2\right\} \\
&= p^2(24N+2)
\end{aligned}$$

by (9). Hence by (10), the integer pair $(\phi(m), \phi(n))$ belongs to the set $S\left(p^2N + \frac{p^2-1}{12}\right)$. Thus ϕ is an injective mapping.

Next we show that ϕ maps $S(N)$ onto $S\left(p^2N + \frac{p^2-1}{12}\right)$. Suppose now $(m,n) \in S\left(p^2N + \frac{p^2-1}{12}\right)$. By (10) we have

$$(6m+1)^2 + (6n+1)^2 = p^2(24N+2). \tag{14}$$

Therefore

$$(6m+1)^2 + (6n+1)^2 \equiv 0 \pmod{p}. \tag{15}$$

It follows from Lemma 1 below that both $6m+1$ and $6n+1$ are divisible by p. Since p^* is of the form $6k+1$, it is easy to check that we can write $6m+1 = p^*(6m'+1)$ and $6n+1 = p^*(6n'+1)$ for some integers m' and n'. One easily checks from (11) that $\phi(m', n') = (m, n)$. From (14), we have $(6m'+1)^2 + (6n'+1)^2 = 24N + 2$, and by (9) this is the condition that $(m', n') \in S(N)$. Thus $\phi : S(N) \to S\left(p^2 N + \frac{p^2-1}{12}\right)$ is an onto map and the proof of part (i) of the theorem is complete.

(ii) Let now $N = pn + \dfrac{p^2-1}{12}$, where $n > 0$ is coprime to p. We show $a(N) = 0$.

By (6) and (9)

$$a(N) \quad = \sum_{(m,n)\in S(N)} t(m, n),$$

where

$$S(N) \quad = \quad \left\{(m, n) \mid \ (6m+1)^2 + (6n+1)^2 = p(24n + 2p), \quad m, n \in \mathbb{Z}\right\}.$$

We claim $S(N)$ is the empty set and so $a(N) = 0$. Suppose on the contrary there exists a pair $(m, n) \in S(N)$. Then

$$(6m+1)^2 + (6n+1)^2 \quad = p(24n + p). \tag{16}$$

It follows that

$$(6m+1)^2 + (6n+1)^2 \quad \equiv \quad 0 \pmod{p}.$$

Hence, by Lemma 1 below, both $6m+1$ and $6n+1$ are divisible by p. Then by (16), p^2 divides $p(24n + 2p)$, and hence p divides n, contradicting the assumption that n and p are coprime. □

Lemma 1. *Let p be a prime congruent to 3 (mod 4). Then for integer x and y,*

$$x^2 + y^2 \quad \equiv \quad 0 \pmod{p}, \tag{17}$$

iff

$$x \equiv 0 \pmod{p} \text{ and } y \equiv 0 \pmod{p}.$$

Proof. The reverse implication is immediate. To prove the forward implication we recall the result from elementary number theory that -1 is a quadratic nonresidue of all primes $\equiv 3 \pmod 4$. Suppose y is not divisible by p. Then $z = x/y$ in $\mathbb{Z}_p$ would give a solution to the congruence

$$z^2 + 1 \quad \equiv \quad 0 \pmod{p},$$

that is to say, -1 is a quadratic residue mod p, a contradiction. Thus we must have $y \equiv 0 \pmod{p}$. It then follows from (17) that $x^2 \equiv 0 \pmod{p}$ and hence also $x \equiv 0 \pmod{p}$. □

By making use of Jacobi's identity $E(q)^3 = \sum_{m\geq 0}(-1)^m(2m+1)q^{m(m+1)/2}$, a proof along the lines of that in Theorem 1 can be given for the following result.

Theorem 2. *Let p be a prime of the form $4k+3$. Let $(E(q)E(-q))^3 = \sum_{n\geq 0} a_3(n)q^n$. Then*

(i)

$$a_3\left(p^2n + \frac{p^2-1}{4}\right) = \epsilon p^2 a(n) \tag{18}$$

where

$$\epsilon = \begin{cases} 1 & p \equiv 7 \text{ or } 23 \pmod{24} \\ -1 & p \equiv 11 \text{ or } 19 \pmod{24} \end{cases} \tag{19}$$

(ii) *If $n > 0$ is coprime to p then*

$$a_3\left(pn + \frac{p^2-1}{4}\right) = 0. \tag{20}$$

□

References

[1] S. Cooper, M. D. Hirschhorn and R. Lewis, Powers of Euler's Product and Related Identities The Ramanujan Journal, Vol. 4 (2), 137-155 (2000)

[2] Wikipedia Pentagonal number theorem