Connected gear wheels

Fig. 1 The game „Mecanix"

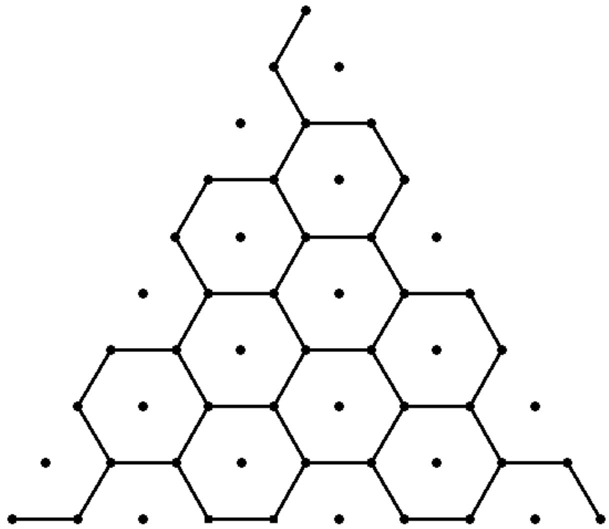


Fig. 2 Connected gear wheels as a graph

Description:

The gear wheels on the left side block each other, they cannot rotate. The graph on the right side shows a system of 37 connected wheels without blockade. Any edge represents the connection of two nodes (wheels). Isolated points are free axles. The example triangle has 10 points on the basic side (n = 10)

Main condition

for a connected and unblocked system of wheels: Any closed polygon has an even number of nodes. Only then nodes can alternately be marked with „+" and "-„, corresponding to the sense of rotation. Obviously the condition is necessary and sufficient.

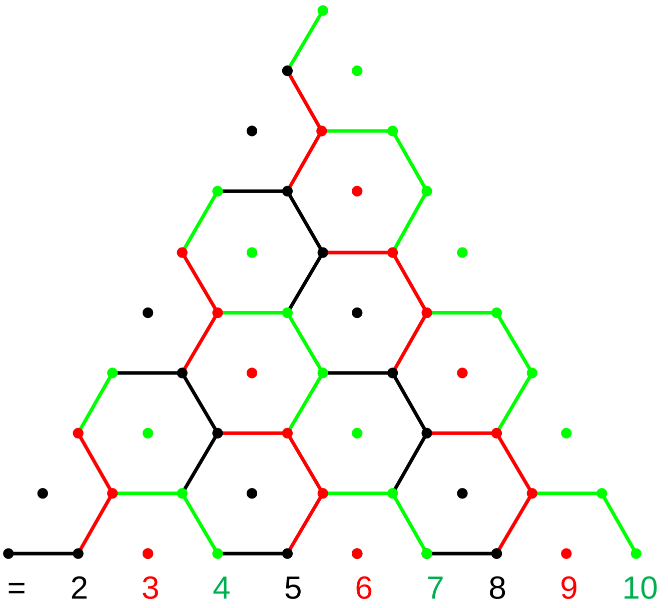
Realization of this condition

Running through a closed polygon, the new direction is, at any node: 0° (straight forward) or 60° (right or left). 180° (backward) makes no sense and 120° causes a triple blockade. Let z(0), z(60) and z(-60) be the number of the corresponding nodes. We select z(0)=0. Then z(60) + z(-60) is even because z(60) = z(-60) ± 6 (6·60°=360°). Therefore, if no other than ±60°-turns occur, the system is unblocked. Example in fig. 2.

Maximum property

The triangle in fig. 2 can be thought of as a part of an infinite graph with a hexagonal pattern. The density of nodes is maximal because the 6 neighbors of any isolated point are all nodes of the graph. The triangle cuts off some half-hexagons, but the maximum property remains: The number of nodes in the intersected graph is maximal.

Fig. 3: Transition n-1→n



From left to right: A new color marks additional points and edges for n-1→n.
n mod 3 = 0: red, n mod 3 = 1: green, n mod 3 = 2: black

<u>Number of nodes b(n):</u>

b(1)=1 , a single wheel is always unblocked

b(n) = 1, 2, 4, 7, 10, 14, 19, 24, 30, 37,.., see fig. 3

= A007980(n) = $\lceil n(n+1)/3 \rceil$

<u>Number of half-hexagons which are not part of a full hexagon</u>

These half-hexagons are associated with isolated points situated on the triangle sides, but not in a corner. Their number on the right side is

$h_{right}(n) = \lfloor (n-1)/3 \rfloor$ = A002264(n-1), n>0

The total number is h(n) = n – 2 + (n mod 3) mod 2, n>1

= A130481(n-2)

<u>Number H(n) of full hexagons</u>

The transition n-1→n converts each half-hexagon on the right side into a full hexagon. Thus H(n) = H(n-1) + h_{right}(n-1)

= H(n-1) + $\lfloor (n-2)/3 \rfloor$ for n>1 with H(1) = 0

Or explicitly: H(n) = $\lfloor (n-2) * (n-3)/6 \rfloor$ for n≥1

= A001840(n-4) for n>3

<u>Number of all half-hexagons</u>

C(n) = h(n) + 6*H(n)

Explicit formula: C(n) = $(n-2)^2$ - (n mod 3) mod 2, n>0

= a(n-1) = A109340(n-1)

C(n) is the maximum number of half-hexagons in a connected system of unblocked gear wheels on a n-triangle.