A FEW IDENTITIES INVOLVING JACOBSTHAL POLYNOMIALS

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ABSTRACT:
The main purpose of this paper is to investigate how to calculate the summation of Jacobsthal polynomials

$$\sum_{a_1+a_2+...+a_k=n} J_{a_{1+1}}(x) \cdot J_{a_{2+1}}(x) \cdot \cdots \cdot J_{a_{k+1}}(x),$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1, a_2, ..., a_k)$ such that $a_1 + a_2 + ... + a_k = n$.

AMS Subj. Classification:
Key Words: Jacobsthal Polynomials and Convolution properties

1. INTRODUCTION

The Jacobsthal polynomials $J(x) = \{J_n(x)\}$, $n = 0, 1, 2, ...$are defined by the second-order linear recurrence sequence

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) \tag{1.1}$$

where $J_1(x) = 1 = J_2(x)$. Also, $J_n(1) = F_n$ where F_n represents the n^{th} Fibonacci number.The first 5 Jacobsthal polynomials are as follows:

$$\begin{aligned} J_1(x) &= 1 \\ J_2(x) &= 1 \\ J_3(x) &= x+1 \\ J_4(x) &= 2x+1 \\ J_5(x) &= x^2+3x+1 \end{aligned}$$

The main purpose of this paper is to investigate how to calculate the summation of Jacobsthal polynomials

$$\sum_{a_1+a_2+...+a_k=n} J_{a_{1+1}}(x) \cdot J_{a_{2+1}}(x) \cdot \cdots \cdot J_{a_{k+1}}(x), \tag{1.2}$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1, a_2, ..., a_k)$ such that $a_1 + a_2 + ... + a_k = n$. A similar summation has been done for Fibonacci polynomials by Yuan and Zhang [2]. However, to the authors' knowledge, nothing similar has been done for Jacobsthal polynomials. This problem will enable us to discover some new convolution properties of $J(x)$ similar to those in Yuan and Zang [2]. In this paper the generating function of the sequence $J(x)$ and its partial derivative is used to derive an expression for (1.2) for any fixed positive integers k and n.

2. Preliminary Results

The following formulas and their proofs can be found in Koshy [1].

The Jacobsthal polynomials can be found explicitly by the formula

$$J_n(x) = \sum_{m=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{\left\lfloor \frac{n}{2} \right\rfloor + m}{\left\lfloor \frac{n-1}{2} \right\rfloor - m} x^{\left\lfloor \frac{n-1}{2} \right\rfloor - m} \tag{2.1}$$

where $\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n-1}{2} & , \text{ if } n \text{ is odd} \\ \frac{n}{2} & , \text{ if } n \text{ is even} \end{cases}$.

Binet's formula for $J_n(x)$ is given by

$$J_n(x) = \frac{r^n - s^n}{\sqrt{1+4x}} \tag{2.2}$$

where $n \geq 1$, $r = \dfrac{1+\sqrt{1+4x}}{2}$ and $s = \dfrac{1-\sqrt{1+4x}}{2}$.

A generating function for $J_n(x)$ is

$$G(t,x) = \frac{1}{1-t-xt^2} = \sum_{n=0}^{\infty} J_{n+1}(x) t^n \tag{2.3}$$

3. Main Results

Proposition 1 *Let* $J(x) = \{J_n(x)\}$ *be defined by* (1.1). *Then for any positive integers* k *and* n, $\sum\limits_{a_1+a_2+\ldots+a_k=n} J_{a_1+1}(x)\cdot J_{a_2+1}(x)\cdot\cdots\cdot J_{a_k+1}(x) = \sum\limits_{m=0}^{\lfloor\frac{n}{2}\rfloor} \binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1} x^{\lfloor\frac{n}{2}\rfloor-m}.$

Proof. From (2.3),

$$G(t,x) = \frac{1}{1-t-xt^2} = \frac{1}{(1-rt)(1-st)} = \frac{1}{r-s}\sum_{n=0}^{\infty}\left(r^{n+1}-s^{n+1}\right)t^n = \sum_{n=0}^{\infty} J_{n+1}(x)t^n. \tag{3.1}$$

Let $\dfrac{\partial G^k(t,x)}{\partial x^k}$ denote the k^{th} partial derivative of $G(t,x)$ with respect to x and $J_n^{(k)}(x)$ denote the k^{th} derivative of $J_n(x)$. Then from (3.1) we have

$$\begin{aligned}
\frac{\partial G(t,x)}{\partial x} &= \frac{t^2}{(1-t-xt^2)^2} = \sum_{n=0}^{\infty} J_{n+1}^{(1)}(x)t^n,\\
\frac{\partial G^2(t,x)}{\partial x^2} &= \frac{2t^4}{(1-t-xt^2)^3} = \sum_{n=0}^{\infty} J_{n+1}^{(2)}(x)t^n,\\
&\vdots\\
\frac{\partial G^{k-1}(t,x)}{\partial x^{k-1}} &= \frac{(k-1)\,t^{2k-2}}{(1-t-xt^2)^k} = \sum_{n=0}^{\infty} J_{n+1}^{(k-1)}(x)t^n = \sum_{n=0}^{\infty} J_{n+2k-1}^{(k-1)}(x)t^{n+2k-2}
\end{aligned} \tag{3.2}$$

using the fact that $J_{n+1}(x)$ is a polynomial of degree $\lfloor\frac{n}{2}\rfloor$.

It is a known fact that for any two absolutely convergent power series $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=0}^{\infty} b_n x^n$, we have $\left(\sum\limits_{n=0}^{\infty} a_n x^n\right)\cdot\left(\sum\limits_{n=0}^{\infty} b_n x^n\right) = \sum\limits_{n=0}^{\infty}\left(\sum\limits_{u+v=n} a_u b_v\right)x^n$. Using this along with (3.2),

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\ldots+a_k=n} J_{a_{1+1}}(x)\cdot J_{a_{2+1}}(x)\cdot\cdots\cdot J_{a_{k+1}}(x)\right)t^n \\
&=\left(\sum_{n=0}^{\infty} J_{n+1}(x)t^n\right)^k \\
&=\tfrac{1}{(1-t-xt^2)^k} \\
&=\tfrac{1}{(k-1)!t^{2k-2}}\frac{\partial G^{k-1}(t,x)}{\partial x^{k-1}} \\
&=\tfrac{1}{(k-1)!}\sum_{n=0}^{\infty} J^{(k-1)}_{n+2k-1}(x)t^n
\end{aligned}
\tag{3.3}
$$

We would like to obtain an expression for the latter part of (3.3). To this end, we take the $(k-1)^{th}$ derivative of equation (2.1) and replace n with $n+2k-1$ to obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\ldots+a_k=n} J_{a_{1+1}}(x)\cdot J_{a_{2+1}}(x)\cdot\cdots\cdot J_{a_{k+1}}(x)\right)t^n \\
&=\tfrac{1}{(k-1)!}\sum_{n=0}^{\infty} J^{(k-1)}_{n+2k-1}(x)t^n \\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}x^{\lfloor\frac{n}{2}\rfloor-m}\right)t^n.
\end{aligned}
\tag{3.4}
$$

Equating coefficients of t^n results in the following identity

$$
\begin{aligned}
&\sum_{a_1+a_2+\ldots+a_k=n} J_{a_{1+1}}(x)\cdot J_{a_{2+1}}(x)\cdot\cdots\cdot J_{a_{k+1}}(x) \\
&=\sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{\left\lfloor\frac{n+2k-1}{2}\right\rfloor+m}{\left\lfloor\frac{n+2k-2}{2}\right\rfloor-m}\binom{\left\lfloor\frac{n+2k-2}{2}\right\rfloor-m}{k-1}x^{\left\lfloor\frac{n}{2}\right\rfloor-m},
\end{aligned}
\tag{3.5}
$$

and this completes the proof. ■

The following corallaries folllow from the proposition and are equivalent to those in Yuan and Zhang [2].

Corollary 2 *1: For any positive integers k and n, we have the identity*

$$\sum_{a_1+a_2+\ldots+a_k=n+k} J_{a_1}(1)\cdot J_{a_2}(1)\cdot\cdots\cdot J_{a_k}(1)$$
$$=\sum_{m=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{\left\lfloor \frac{n+2k-1}{2}\right\rfloor+m}{\left\lfloor \frac{n+2k-2}{2}\right\rfloor-m}\binom{\left\lfloor \frac{n+2k-2}{2}\right\rfloor-m}{k-1}.$$

Proof. Let $x=1$ in the proposition and note that, Yuan and Zhang [2],

$$\sum_{a_1+a_2+\ldots+a_k=n} J_{2a_1+1}(1)\cdot J_{2a_2+1}(1)\cdot\cdots\cdot J_{2a_k+1}(1)$$
$$=\sum_{a_1+a_2+\ldots+a_k=n} F_{2a_1+1}\cdot F_{a_2+1}\cdot\cdots\cdot F_{2a_k+1}$$
$$=\sum_{a_1+a_2+\ldots+a_k=n+k} F_{2a_1}\cdot F_{a_2}\cdot\cdots\cdot F_{2a_k}$$
$$=\sum_{a_1+a_2+\ldots+a_k=n+k} J_{2a_1}(1)\cdot J_{2a_2}(1)\cdot\cdots\cdot J_{2a_k}(1),$$

we obtain the desired result which is equivalent to corollary 1 in Yuan and Zhang [2]. ■

Corollary 3 *2: For any positive integers k and n, we have the identity*

$$\sum_{a_1+a_2+\ldots+a_k=n+k} J_{2a_1}(1)\cdot J_{2a_2}(1)\cdot\cdots\cdot J_{2a_k}(1)$$
$$=(-1)^{\left\lfloor \frac{n}{2}\right\rfloor}\cdot(3)^{n-2\left\lfloor \frac{n}{2}\right\rfloor}\sum_{m=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{\left\lfloor \frac{n+2k-1}{2}\right\rfloor+m}{\left\lfloor \frac{n+2k-2}{2}\right\rfloor-m}\binom{\left\lfloor \frac{n+2k-2}{2}\right\rfloor-m}{k-1}(-1)^m\cdot 3^{2m}.$$

Proof. Let $x=-\frac{1}{9}$ in the proposition and note that

$$\sum_{a_1+a_2+\ldots+a_k=n} J_{2a_1+1}(1)\cdot J_{2a_2+1}(1)\cdot\cdots\cdot J_{2a_k+1}(1)$$
$$=\sum_{a_1+a_2+\ldots+a_k=n} F_{2a_1+1}\cdot F_{a_2+1}\cdot\cdots\cdot F_{2a_k+1}$$
$$=\sum_{a_1+a_2+\ldots+a_k=n+k} F_{2a_1}\cdot F_{a_2}\cdot\cdots\cdot F_{2a_k}$$
$$=\sum_{a_1+a_2+\ldots+a_k=n+k} J_{2a_1}(1)\cdot J_{2a_2}(1)\cdot\cdots\cdot J_{2a_k}(1),$$

as seen in Yuan and Zhang [2]. From (2.2),

$$
\begin{aligned}
J_n\left(-\tfrac{1}{9}\right) &= \frac{\left(\frac{3+\sqrt{5}}{6}\right)^n - \left(\frac{3-\sqrt{5}}{6}\right)^n}{\frac{\sqrt{5}}{3}} \\
&= \frac{3^n\left(\frac{3+\sqrt{5}}{6}\right)^n - \left(\frac{3-\sqrt{5}}{6}\right)^n}{3^n\frac{\sqrt{5}}{3}} \\
&= \frac{\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n}{3^{n-1}\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}}{3^{n-1}\sqrt{5}} \\
&= \frac{1}{3^{n-1}}J_{2n}(1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{a_1+a_2+\ldots+a_k=n+k} J_{2a_1}(1)\cdot J_{2a_2}(1)\cdot\dots\cdot J_{2a_k}(1) \\
&= \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor+m}{k-1}\left(-\frac{1}{9}\right)^{\lfloor\frac{n}{2}\rfloor-m} \\
&= (-1)^{\lfloor\frac{n}{2}\rfloor}\cdot(3)^{n-2\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}(-1)^m\cdot 3^{2m}
\end{aligned}
$$

■

Corollary 4 *3: For any positive integers k and n, we have the identity*

$$
\begin{aligned}
&\sum_{a_1+a_2+\ldots+a_k=n+k} J_{3a_1}(1)\cdot J_{3a_2}(1)\cdot\dots\cdot J_{3a_k}(1) \\
&= 2^{2n+k-4\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}2^{4m}.
\end{aligned}
$$

Proof. Let $x=\frac{1}{16}$ in the proposition and note that $\sum_{a_1+a_2+\ldots+a_k=n} J_{3a_1+1}(1)\cdot J_{3a_2+1}(1)\cdot\dots\cdot J_{3a_k+1}(1) = \sum_{a_1+a_2+\ldots+a_k=n+k} J_{3a_1}(1)\cdot J_{3a_2}(1)\cdot\dots\cdot J_{3a_k}(1)$, as seen in

Yuan and Zhang [2]. From (2.2), we see that

$$\begin{aligned} J_n\left(\frac{1}{16}\right) &= \frac{\left(\frac{2+\sqrt{5}}{4}\right)^n - \left(\frac{2-\sqrt{5}}{4}\right)^n}{\frac{\sqrt{5}}{2}} \\ &= \frac{4^n\left(\frac{2+\sqrt{5}}{4}\right)^n - \left(\frac{2-\sqrt{5}}{4}\right)^n}{4^n\frac{\sqrt{5}}{2}} \\ &= \frac{\left(2+\sqrt{5}\right)^n - \left(2-\sqrt{5}\right)^n}{2^{n-1}\sqrt{5}} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{3n} - \left(\frac{1-\sqrt{5}}{2}\right)^{3n}}{2^{n-1}\sqrt{5}} \\ &= \frac{1}{2^{n-1}} J_{3n}(1). \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{a_1+a_2+\ldots+a_k=n+k} J_{3a_1}(1)\cdot J_{3a_2}(1)\cdot\cdots\cdot J_{3a_k}(1) \\ &= \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor} \binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}\left(\frac{1}{16}\right)^{\lfloor\frac{n}{2}\rfloor-m} \\ &= 2^{2n+k-4\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}2^{4m}. \end{aligned}$$

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Corollary 5 *4: For any positive integers k and n, we have the identity*

$$\begin{aligned} &\sum_{a_1+a_2+\ldots+a_k=n+k} J_{4a_1}(1)\cdot J_{4a_2}(1)\cdot\cdots\cdot J_{4a_k}(1) \\ &= (-1)^{\lfloor\frac{n}{2}\rfloor}3^k\cdot 7^{n-2\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}7^{2m}(-1)^m. \end{aligned}$$

Proof. Let $x=-\frac{1}{49}$ in the proposition and note that $\sum_{a_1+a_2+\ldots+a_k=n} J_{4a_1+1}(1)\cdot J_{4a_2+1}(1)\cdot\cdots\cdot J_{4a_k+1}(1) = \sum_{a_1+a_2+\ldots+a_k=n+k} J_{4a_1}(1)\cdot J_{4a_2}(1)\cdot\cdots\cdot J_{4a_k}(1)$, as seen in

Yuan and Zhang [2]. From (2.2), we see that

$$J_n\left(-\frac{1}{49}\right) = \frac{\left(\frac{7+3\sqrt{5}}{14}\right)^n - \left(\frac{7-3\sqrt{5}}{14}\right)^n}{\frac{3\sqrt{5}}{7}}$$

$$= \frac{7^n\left(\frac{7+3\sqrt{5}}{14}\right)^n - \left(\frac{7-3\sqrt{5}}{14}\right)^n}{7^{n-1}3\sqrt{5}}$$

$$= \frac{\left(\frac{7+3\sqrt{5}}{2}\right)^n - \left(\frac{7-3\sqrt{5}}{2}\right)^n}{7^{n-1}3\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{4n} - \left(\frac{1-\sqrt{5}}{2}\right)^{4n}}{7^{n-1}3\sqrt{5}}$$

$$= \frac{1}{3}\frac{1}{7^{n-1}}J_{3n}(1).$$

Therefore,

$$\sum_{a_1+a_2+\ldots+a_k=n+k} J_{4a_1}(1)\cdot J_{4a_2}(1)\cdot\cdots\cdot J_{4a_k}(1)$$

$$= \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}\left(-\frac{1}{49}\right)^{\lfloor\frac{n}{2}\rfloor-m}$$

$$= (-1)^{\lfloor\frac{n}{2}\rfloor}\cdot 3^k\cdot 7^{n-2\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n+2k-1}{2}\rfloor+m}{\lfloor\frac{n+2k-2}{2}\rfloor-m}\binom{\lfloor\frac{n+2k-2}{2}\rfloor-m}{k-1}(-1)^m\cdot 7^{2m}.$$

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4. References

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