

A103345 Comments by Wolfdieter Lang

Added (Dec 02 2009): See the link under A164655 for $\Theta(k,n) := \sum(1/(2^m - 1)^k, m=1..n)$.

Added (Dec 03 2013): See a note at the end of this file for more formulae involving the Polygamma function.

The rational sequences $Zeta(k,n) := \sum(1/m^k, m=1..n)$, the partial sums of Euler's Zeta(k) series, obtained in the limit $n \rightarrow \infty$, are, for $k=2..10$, and $n=1..20$, given below.

For $n=1$: $Zeta(1,n) = H(n)$, the harmonic numbers, and the series diverges.

For $n \geq 2$ one has (See Abramowitz-Stegun, p.260, 6.4.3.)

$$Zeta(k,n) = (Zeta(k) + ((-1)^{(k+1)}) * Psi(k-1, n+1) / (k-1)!) / (k-1)!$$

with the Zeta(k) = Zeta(1,k) function and the Polygamma function Psi(n,x) (the n-th derivative of the logarithmic derivative of the Gamma function).

The g.f.s for the sequences $Zeta(k,n)$, $n=1..infty$, are $G(k;x) = \text{polylog}(k,x)/(1-x)$ with the polylogarithm functions. $\text{polylog}(k,x)$ is, per definition, the g.f. for the sequence of the k-th powers of the inverse natural numbers. For the inverse numbers ($k=1$) this is the $\text{polylog}(1,x) = -\ln(1-x)$ function, and for $k=2$ it is the dilogarithm function $\text{dilog}(1-x)$ (Maple9).

$k=1$: Harmonic numbers A001008/A002805. G.f. $\text{polylog}(1,x)/(1-x) = -\ln(1-x)/(1-x)$.

[1, 3/2, 11/6, 25/12, 137/60, 49/20, 363/140, 761/280, 7129/2520, 7381/2520, 83711/27720, 86021/27720, 1145993/360360, 1171733/360360, 1195757/360360, 2436559/720720, 42142223/12252240, 14274301/4084080, 275295799/77597520, 55835135/15519504].

Zeta(1) does not exist (is infinite).

$k=2$: A007406/A007407; $Zeta(2) = (\pi^2)/6 = 1.6449340668\dots$

G.f.: $\text{polylog}(2,x)/(1-x)$.

[1, 5/4, 49/36, 205/144, 5269/3600, 5369/3600, 266681/176400, 1077749/705600, 9778141/6350400, 1968329/1270080, 239437889/153679680, 240505109/153679680, 40799043101/25971865920, 40931552621/25971865920, 205234915681/129859329600, 822968714749/519437318400, 238357395880861/150117385017600, 238820721143261/150117385017600, 86364397717734821/54192375991353600, 17299975731542641/10838475198270720].

Values for $n=10^m$, $m=0..5$: 1., 1.549767731, 1.634983900, 1.643934568, 1.644834073, 1.644924068 (Maple9, digits=10).

k=3: A007408/A007409; Zeta(3)= 1.2020569031... G.f.: polylog(3,x)/(1-x).

[1, 9/8, 251/216, 2035/1728, 256103/216000, 28567/24000, 9822481/8232000, 78708473/65856000, 19148110939/16003008000, 19164113947/16003008000, 25523438671457/21300003648000, 25535765062457/21300003648000, 56123375845866029/46796108014656000, 56140429821090029/46796108014656000, 56154295334575853/46796108014656000, 449325761325072949/374368864117248000, 2207911834254200646437/1839274229408039424000, 245358578943756786493/204363803267559936000, 1683118856778495358491487/1401731326612193601024000, 336658814638864376538323/280346265322438720204800].

Values for n=10^m, m=0..5: 1., 1.197531986, 1.202007401, 1.202056403, 1.202056898, 1.202056903 (Maple9,digits=10).

k=4: A007410/A007480; Zeta(4)= (Pi^4)/90 = 1.0823232337...
G.f.: polylog(4,x)/(1-x).

[1, 17/16, 1393/1296, 22369/20736, 14001361/12960000, 14011361/12960000, 33654237761/31116960000, 538589354801/497871360000, 43631884298881/40327580160000, 43635917056897/40327580160000, 638913789210188977/590436101122560000, 638942263173398977/590436101122560000, 18249420414596570742097/16863445484161436160000, 18249859383918836502097/16863445484161436160000, 18250192489014819937873/16863445484161436160000, 292007196876357275606593/269815127746582978560000, 24389002905437982598916813953/22535229284522356952309760000, 24389217575736760627770573953/22535229284522356952309760000, 3178449758916874904128641277888913/2936813615588238080381960232960000, 3178468114001972330616643665140369/2936813615588238080381960232960000].

Values for n=10^m, m=0..5: 1., 1.082036583, 1.082322905, 1.082323234, 1.082323234, 1.082323234 (Maple9,digits=10).

k=5: A099828/A069052; Zeta(5)= 1.0369277551... G.f.: polylog(5,x)/(1-x).

[1, 33/32, 8051/7776, 257875/248832, 806108207/777600000, 268736069/259200000, 4516906311683/4356374400000, 144545256245731/139403980800000, 105375212839937899/101625502003200000, 105376229094957931/101625502003200000, 16971048697474072945481/16366888723117363200000, 16971114472329088045481/16366888723117363200000, 6301272372663207205033976933/6076911214672415134617600000, 6301283671733562325696376933/6076911214672415134617600000, 6301291674250388231757459557/6076911214672415134617600000, 201641519028625175870313858949/194461158869517284307763200000, 286302316744585525852980532588950293/276107037648996202745367733862400000, 95434154288788166027220236648583431/92035679216332067581789244620800000, 236304506035993305443901582534565405715669/227889453271880616207200766816318259200000, 1890436618011579623252753178278440286521/1823115626175044929657606134530546073600].

Values for n=10^m, m=0..5: 1., 1.036907341, 1.036927753, 1.036927755, 1.036927755, 1.036927755 (Maple9,digits=10).

k=6: A103345/A103346; Zeta(6)= (Pi^6)/945 = 1.0173430619...

[1, 65/64, 47449/46656, 3037465/2985984, 47463376609/46656000000,
47464376609/46656000000, 5584183099672241/5489031744000000,
357389058474664049/351298031616000000, 260537105518334091721/256096265048064000000,
52107472322919827957/51219253009612800000,
92311616995117182948130877/90738031080962661580800000,
92311647383100199924330877/90738031080962661580800000,
445570781131605573859221176881493/437975145063870303582159667200000,
445570839299219762020391212081493/437975145063870303582159667200000,
2227854388748563035669167509366441/2189875725319351517910798336000000,
142582689233620975717120535871967849/140152046420438497146291093504000000,
3441599639734129841657818562217692625019081/
3382929690964537235124904363538251776000000,
3441599739196302352474436585340919249019081/
3382929690964537235124904363538251776000000,
161913095162789967079069634260899594958212827455361/
159152907672484393983755270823401332001734656000000,
32382619529910829892327658051415140314771727996493/
31830581534496878796751054164680266400346931200000].

Values for n=10^m, m=0..5: 1., 1.017341512, 1.017343062, 1.017343063, 1.017343063,
1.017343063 (Maple9, digits=10).

k=7: A103347/A103348, Zeta(7)= 1.0083492773...

[1, 129/128, 282251/279936, 36130315/35831808, 2822716691183/2799360000000,
940908897061/933120000000, 774879868932307123/768464444160000000,
99184670126682733619/98363448852480000000,
650750755630450535274259/645362587921121280000000,
650750820166709327386387/645362587921121280000000,
12681293156341501091194786541177/12576291107821424895098880000000,
12681293507322704937269896541177/12576291107821424895098880000000,
795732373802519483063688931801166184509/789143616376081512994335288360960000000,
795732381288691429080031515331406184509/789143616376081512994335288360960000000,
795732385907364629635058203181196623933/789143616376081512994335288360960000000,
101853745772435672040195214916143634191549/
1010103828961384336632749169102028800000000,
41794530981350996538975340812915102041760524474477/
41448466476823341693686758239117909939978240000000,
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138161554922744472312289194130393033133260800000000,
12452983396984961254768954838142092557598337617274512368501/
12349870936173761211842329302824301328031245003653120000000,
2490596681326659584730941156978782465085964605959784505521/
2469974187234752242368465860564860265606249000730624000000].

Values for n=10^m, m=0..5: 1., 1.008349155, 1.008349277, 1.008349277, 1.008349277,
1.008349277 (Maple9, digits=10).

k=8: A103349/A103350; Zeta(8)= (Pi^8)/9450 = 1.0040773561...

[1, 257/256, 1686433/1679616, 431733409/429981696, 168646292872321/1679616000000000,
168646392872321/1679616000000000, 972213062238348973121/968265199641600000000,
248886558707571775009601/247875891108249600000000,

1632944749460578249437992161/1626313721561225625600000000,
1632944765723715465050248417/1626313721561225625600000000,
350036214341656523811791485040141377/348614789508809898092140953600000000,
350036215152423104696224989140141377/348614789508809898092140953600000000,
285535294511012013599719994461645649102142817/
284375793597284734022638664513755545600000000,
285535294703706079490180652561714026702142817/
284375793597284734022638664513755545600000000,
285535294814665084460314613710617392218864993/
284375793597284734022638664513755545600000000,
73097035489504379781926501380241726163779828833/
72800203160904891909795498115521419673600000000,
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509907189349834795004216817611957628024292411478706096353/
507836558905994020033056646767650020882998991257600000000,
8660040895889879602795422202778138097739048470649516007779795689473/
8624874212704459432000434464302453009091379853162857470361600000000,
8660040896226788751729190149340655069000863041317148033293969809409/
8624874212704459432000434464302453009091379853162857470361600000000].

Values for $n=10^m$, $m=0..5$: 1., 1.004077346, 1.004077356, 1.004077357, 1.004077357,
1.004077357 (Maple9, digits=10).

k=9: A103351/A103352; Zeta(9)= 1.0020083928...

[1, 513/512, 10097891/10077696, 5170139875/5159780352,
10097934603139727/100776960000000000, 373997614931101/3732480000000000,
15092153145114981831307/150619031055360000000000,
7727182467755471289426059/77116943900344320000000000,
4106541588424891370931874221019/40983105783342885765120000000000,
4106541592523201949266162797531/40983105783342885765120000000000,
9683010270183857478533136289380321950921/96636019651842103751141472337920000000000,
9683010272056728280376177683851321950921/96636019651842103751141472337920000000000,
102683476368441736434864754827652083087554881272533/
1024776609807175267523980691441769484124160000000000,
102683476373401681690885212167147843126978881272533/
1024776609807175267523980691441769484124160000000000,
102683476376067360826287710449789097580152614828757/
1024776609807175267523980691441769484124160000000000,
52573939905310006815580379830668747845365915138026709/
524685624221273736972278114018185975871569920000000000,
6234631892504071525166401712432923053805706595748851067179358373/
62221354004903761720098179697924722918635155606460170240000000000,
230912292316127390041346190264217706734259982976273801302939199/
2304494592774213397040673322145360108097598355794821120000000000,
74512555996593068624603395097996982840739874462671285157977822783324339021/
74363205468646504984871372594710987046892058887044655090392629248000000000,
596100447973906474082274762424364477922711354873977969683932655002382097/
5949056437491720398789709807576878963751364710963572407231410339840000000].

Values for $n=10^m$, $m=0..5$: 1., 1.002008392, 1.002008393, 1.002008393, 1.002008393,
1.002008393 (Maple9, digits=10).

k=10: A103716/A103717; Zeta(10)= (Pi^10)/93555 = 1.0009945751...

[1, 1025/1024, 60526249/60466176, 61978938025/61917364224, 605263128567754849/604661760000000000, 605263138567754849/604661760000000000, 170971856382109814342232401/170801981216778240000000000, 175075181098169912564190119249/174901228765980917760000000000, 10338014371627802833957102351534201/10327742657402407212810240000000000, 413520574906423083987893722912609/413109706296096288512409600000000, 10725658733010631077837980123359391649694009/, 107150018589962524639265664528285696000000000, 10725658733183684339928277148208512049694009/, 107150018589962524639265664528285696000000000, 1478623137054473615254787592944842689807988392270632641/, 1477153996440454717619766727871824205195929190400000000, 1478623137059580374890386255821587524344579342670632641/, 1477153996440454717619766727871824205195929190400000000, 36965578426553549647808566014281861605597529341720541401/, 369288499110113679404941681967956051298982297600000000000, 37852752308825227510932683290485196081771414731414031660249/, 37815142308875640771066028233518699653015787274240000000000, 76310917769836275757123875617429924166816445033269207203147996696551801/, 76235096239304206549745572122210120713261839892769555622133760000000000, 76310917769857627331413884469115370235288630968406710637654236696551801/, 76235096239304206549745572122210120713261839892769555622133760000000000, 46786729304067707085560788686890187212152389766530307188847138901970307271761684960 1/467402426213009930022986138619988554638866745663618654505857352329865461760000000 000, 18714691721628908624951710044795227174464940236903680948764104071157286413987212521 /1869609704852039720091944554479954218555466982654474618023429409319461847040000000 0].

Values for n=10^m, m=0..5: 1., 1.000994575, 1.000994575, 1.000994577, 1.000994577, 1.000994577 (Maple9,digits=10).

Note added, Dec 03 2013

On Nov 30 2013 Jean-François Alcover has send me (WL) an e-mail with a formula for Theta(k, n) := sum(1/(2*j-1)^k, j=1..n) involving the Polygamma function Psi, allowing for an extension to complex values of n. See the W. Lang link under A164655, the note added at the end. An analog calculation leads to the above given result for Zeta(k, n), for k >= 2, found in the Abramowitz-Stegun handbook on p. 269, eq. 6.4.3. A more general formula, valid for all natural k is

$$Zeta(k, n) = (-1)^k * (Psi(k-1, 1) - Psi(k-1, n+1)) / (k-1)! .$$

The right-hand side is valid also for complex n, not a negative natural number. For k=1 this becomes Zeta(k, n) = gamma + Psi(n+1), with Psi(n+1) = Psi(0, n+1), given in the Abramowitz-Stegun handbook on p. 258, eq. 6.3.2. For k = 2, 3, ... one obtains the result given at the beginning of this file.

Using the recurrence for Psi, like Psi(k-1, 2*j+1) - Psi(k-1, 2*j) = -(-1)^k * (k-1)! / j^k, after summing over j leads to

$$Zeta(k, n) = (-2)^k * sum((-1)^j * Psi(k-1, j), j=2..2*n+1) / (k-1)! .$$

One should not add and subtract the j=1 term because it diverges. Combining the two results one obtains a formula for the alternating sum

$$\sum_{j=2}^{2n+1} (-1)^j \Psi(k-1, j) = (\Psi(k-1, 1) - \Psi(k-1, n+1)) / 2^k, \quad k=1, 2, \dots$$

and the right-hand side is again valid for complex n , not a negative natural number.

This can be split into the cases $k=1$:

$$\sum_{j=2}^{2n+1} (-1)^j \Psi(j) = -(\gamma + \Psi(n+1)) / 2^k,$$

and $k = 2, 3, \dots$

$$\sum_{j=2}^{2n+1} (-1)^j \Psi(k-1, j) = ((-1)^k (k-1)! \zeta(k) - \Psi(k-1, n+1)) / 2^k.$$

#####e.o.f.#####