COUNTING WALKS ON FINITE GRAPHS

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Abstract. This illustrates the Matrix Transfer Method of counting walks along edges of simple finite graphs defined by their Adjacency Matrices.

1. Walks on Labeled Finite Graphs

1.1. **Adjacency Matrices.** The class of walks of n steps on a finite graph considered here starts at one of the vertices, successively steps to a vertex that is adjacent to the current vertex, and finishes at another or the same vertex. Labeling the vertices helps to distinguish the paths if the graph is asymmetric.

The algebra that ensues is explained by the small example of walks on the graph of the square; vertices are labeled clockwise 1 to 4, and each step can be clockwise or counter-clockwise. The number of walks with n steps that start on vertex i and finish on vertex j are recursively bound to the number of walks with $n-1$ steps that finish on a vertex adjacent to j. Let $W^{\square}_{i\to j}(n)$ be the walks that end on vertex j after n steps; the recurrence can be written

$$W^{\square}_{i\to 1}(n+1) = W^{\square}_{i\to 2}(n)+W^{\square}_{i\to 4}(n); \tag{1}$$
$$W^{\square}_{i\to 2}(n+1) = W^{\square}_{i\to 1}(n)+W^{\square}_{i\to 3}(n); \tag{2}$$
$$W^{\square}_{i\to 3}(n+1) = W^{\square}_{i\to 2}(n)+W^{\square}_{i\to 4}(n); \tag{3}$$
$$W^{\square}_{i\to 4}(n+1) = W^{\square}_{i\to 1}(n)+W^{\square}_{i\to 3}(n). \tag{4}$$

where the initial condition, starting at vertex 1, is

$$W^{\square}_{i\to j}(0) = \delta_{i1}\delta_{j1}. \tag{5}$$

Rewritten as a system of linear recurrences in matrix format condenses the information of the structure into the binary Adjacency Matrix A:

$$\begin{pmatrix} W^{\square}_{i\to 1}(n+1)\\ W^{\square}_{i\to 2}(n+1)\\ W^{\square}_{i\to 3}(n+1)\\ W^{\square}_{i\to 4}(n+1)\end{pmatrix} = \begin{pmatrix} 0&1&0&1\\ 1&0&1&0\\ 0&1&0&1\\ 1&0&1&0\end{pmatrix}\cdot\begin{pmatrix} W^{\square}_{i\to 1}(n)\\ W^{\square}_{i\to 2}(n)\\ W^{\square}_{i\to 3}(n)\\ W^{\square}_{i\to 4}(n)\end{pmatrix}. \tag{6}$$

1.2. **Transfer Matrices.** Such systems of linear recurrences with constant coefficients are solved with the transfer matrix method: Define generating functions

$$W^{\square}_{i\to j}(z) \equiv \sum_{n\ge 0} W^{\square}_{i\to j}(n) z^n \tag{7}$$

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such that

$$W^{\square}_{i\to j}(z) = W^{\square}_{i\to j}(0) + \sum_{n\ge 1} W^{\square}_{i\to j}(n)z^n = W^{\square}_{i\to j}(0) + z\sum_{n\ge 0} W^{\square}_{i\to j}(n+1)z^n. \tag{8}$$

Multiplication of each line of the system of linear recurrences by z^n and insertion of that split gives

$$\begin{pmatrix} \sum_n W^{\square}_{i\to 1}(n+1)z^n \\ \sum_n W^{\square}_{i\to 2}(n+1)z^n \\ \sum_n W^{\square}_{i\to 3}(n+1)z^n \\ \sum_n W^{\square}_{i\to 4}(n+1)z^n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sum_n W^{\square}_{i\to 1}(n)z^n \\ \sum_n W^{\square}_{i\to 2}(n)z^n \\ \sum_n W^{\square}_{i\to 3}(n)z^n \\ \sum_n W^{\square}_{i\to 4}(n)z^n \end{pmatrix}. \tag{9}$$

$$\frac{1}{z}\begin{pmatrix} W^{\square}_{i\to 1}(z) - W^{\square}_{i\to 1}(0) \\ W^{\square}_{i\to 2}(z) - W^{\square}_{i\to 2}(0) \\ W^{\square}_{i\to 3}(z) - W^{\square}_{i\to 3}(0) \\ W^{\square}_{i\to 4}(z) - W^{\square}_{i\to 4}(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} W^{\square}_{i\to 1}(z) \\ W^{\square}_{i\to 2}(z) \\ W^{\square}_{i\to 3}(z) \\ W^{\square}_{i\to 4}(z) \end{pmatrix}. \tag{10}$$

$$\begin{pmatrix} W^{\square}_{i\to 1}(z) \\ W^{\square}_{i\to 2}(z) \\ W^{\square}_{i\to 3}(z) \\ W^{\square}_{i\to 4}(z) \end{pmatrix} = \begin{pmatrix} 0 & z & 0 & z \\ z & 0 & z & 0 \\ 0 & z & 0 & z \\ z & 0 & z & 0 \end{pmatrix} \cdot \begin{pmatrix} W^{\square}_{i\to 1}(z) \\ W^{\square}_{i\to 2}(z) \\ W^{\square}_{i\to 3}(z) \\ W^{\square}_{i\to 4}(z) \end{pmatrix} + \begin{pmatrix} W^{\square}_{i\to 1}(0) \\ W^{\square}_{i\to 2}(0) \\ W^{\square}_{i\to 3}(0) \\ W^{\square}_{i\to 4}(0) \end{pmatrix}. \tag{11}$$

The diagonal unit matrix minus z times the adjacency matrix, $1 - zA$, appears in a linear system of equations where the initial values at $n = 0$ are the right hand side:

$$\begin{pmatrix} 1 & -z & 0 & -z \\ -z & 1 & -z & 0 \\ 0 & -z & 1 & -z \\ -z & 0 & -z & 1 \end{pmatrix} \cdot \begin{pmatrix} W^{\square}_{i\to 1}(z) \\ W^{\square}_{i\to 2}(z) \\ W^{\square}_{i\to 3}(z) \\ W^{\square}_{i\to 4}(z) \end{pmatrix} = \begin{pmatrix} W^{\square}_{i\to 1}(0) \\ W^{\square}_{i\to 2}(0) \\ W^{\square}_{i\to 3}(0) \\ W^{\square}_{i\to 4}(0) \end{pmatrix}. \tag{12}$$

Multiplying with the inverse of this matrix of z, the generating functions become uncoupled [4, 1]:

$$\begin{pmatrix} W^{\square}_{i\to 1}(z) \\ W^{\square}_{i\to 2}(z) \\ W^{\square}_{i\to 3}(z) \\ W^{\square}_{i\to 4}(z) \end{pmatrix} = \frac{1}{4z^2-1}\begin{pmatrix} 2z^2-1 & -z & -2z^2 & -z \\ -z & 2z^2-1 & -z & -2z^2 \\ -2z^2 & -z & 2z^2-1 & -z \\ -z & -2z^2 & -z & 2z^2-1 \end{pmatrix} \cdot \begin{pmatrix} W^{\square}_{i\to 1}(0) \\ W^{\square}_{i\to 2}(0) \\ W^{\square}_{i\to 3}(0) \\ W^{\square}_{i\to 4}(0) \end{pmatrix}, \tag{13}$$

and the initial condition (5) provides explicit rational generating functions such that only the first column of the inverse matrix is needed:

$$\begin{pmatrix} W^{\square}_{1\to 1}(z) \\ W^{\square}_{1\to 2}(z) \\ W^{\square}_{1\to 3}(z) \\ W^{\square}_{1\to 4}(z) \end{pmatrix} = \frac{1}{(1+2z)(1-2z)}\begin{pmatrix} 1-2z^2 \\ z \\ 2z^2 \\ z \end{pmatrix}. \tag{14}$$

The technique of splitting the rational generating functions into partial fractions and the Binet formula (rewriting the geometric series as Taylor series) gives in this case

(15)
$$\begin{pmatrix} W^{\square}_{1\to 1}(z) \\ W^{\square}_{1\to 2}(z) \\ W^{\square}_{1\to 3}(z) \\ W^{\square}_{1\to 4}(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{4}\frac{1}{1-2z} + \frac{1}{4}\frac{1}{1+2z} = \frac{1}{2} + \frac{1}{4}\sum_{n\ge 0}[(-2)^n + 2^n]z^n \\ \frac{1}{4}\frac{1}{1-2z} - \frac{1}{4}\frac{1}{1+2z} = \frac{1}{4}\sum_{n\ge 0}[(-2)^n - 2^n]z^n \\ -\frac{1}{2} + \frac{1}{4}\frac{1}{1-2z} + \frac{1}{4}\frac{1}{1+2z} = -\frac{1}{2} + \frac{1}{4}\sum_{n\ge 0}[(-2)^n + 2^n]z^n \\ \frac{1}{4}\frac{1}{1-2z} - \frac{1}{4}\frac{1}{1+2z} = \frac{1}{4}\sum_{n\ge 0}[(-2)^n - 2^n]z^n \end{pmatrix}.$$

We get sequence [3, A199573] for $W^{\square}_{1\to 1}(n)$ and (setting the first element to 0) for $W^{\square}_{1\to 3}(n)$, and [3, A199572] for $W^{\square}_{1\to 2}(n)$ and $W^{\square}_{1\to 4}(n)$.

Remark 1. *For r-regular graphs (graphs where the degree of every vertex is the same, r) the probabilities of reaching vertices after n steps is $W_{i\to j}(n)/r^n$.*

Remark 2. *For r-regular graphs the number of paths with n steps is $\sum_j W_{i\to j}(n) = r^n$, so the sum rule of the generating functions is $\sum_j W_{i\to j}(z) = 1/(1-rz)$.*

Remark 3. *We are only considering undirected graphs, so the adjacency matrices are symmetric. The technique is also applicable to* reluctant/hesitant *random walks with non-zero entries on the diagonal of the adjacency matrix, and potentially also to self-avoiding walks* [5].

2. Tetrahedron

The tetrahedron is the complete graph K_4 of Figure 1 with the adjacency matrix

(16)
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Inverting $1 - zA$ gives

(17)
$$\begin{pmatrix} W^{\square}_{1\to 1}(z) \\ W^{\square}_{1\to 2}(z) \\ W^{\square}_{1\to 3}(z) \\ W^{\square}_{1\to 4}(z) \end{pmatrix} = \begin{pmatrix} \frac{1-2z}{(1+z)(1-3z)} \\ \frac{z}{(1+z)(1-3z)} \\ \frac{z}{(1+z)(1-3z)} \\ \frac{z}{(1+z)(1-3z)} \end{pmatrix},$$

so $W_{1\to 1}(z) = 1 + 3z^2 + 6z^3 + 21z^4 + \cdots$ [3, A054878][2] and $W_{1\to 2}(z) = W_{1\to 3}(z) = W_{1\to 4}(z) = z + 2z^2 + 7z^3 + 20z^4 + \cdots$ [3, A015518].

3. Y_3 Prism

The adjacency matrix of the prism graph of Figure 2 is

(18)
$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Inverting $1 - zA$ gives [3, A094554,A094555,A094556]

(19)
$$W_{1\to 1}(z) = \frac{1 - 2z - 2z^2 + 2z^3}{(1-z)(1+2z)(1-3z)} = 1 + 3z^2 + 2z^3 + 19z^4 + 30z^5 + \cdots$$

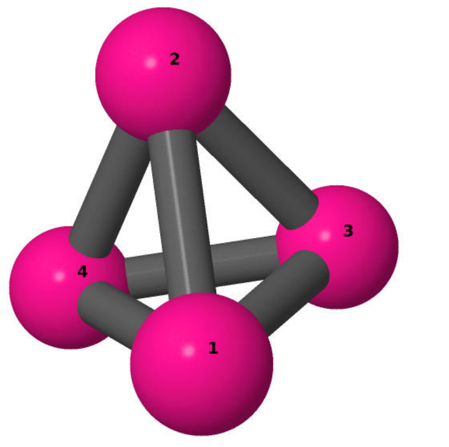

FIGURE 1. Structure of the edges (grey) in the tetrahedron of 4 vertices.

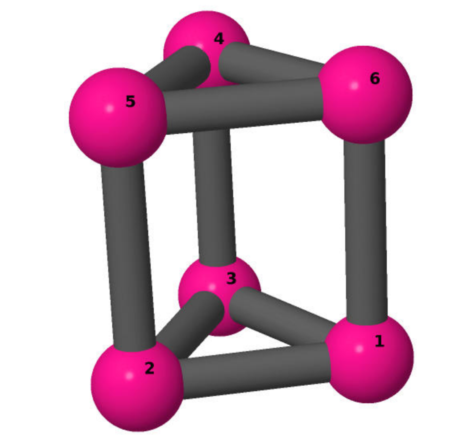

FIGURE 2. The prism graph with triangular lids.

(20)
$$W_{1\to 2}(z) = W_{1\to 3}(z) = \frac{z(1-z-z^2)}{(1-z)(1+2z)(1-3z)} = z+z^2+6z^3+11z^4+46z^5+111z^6+\cdots$$

(21)
$$W_{1\to 4}(z) = W_{1\to 5}(z) = \frac{z^2(2-z)}{(1-z)(1+2z)(1-3z)} = 2z^2+3z^3+16z^4+35z^5+132z^6+\cdots$$

(22)
$$6W_{1\to 4}(n) = 6W_{1\to 5}(n) = 3^n + (-2)^n - 1;$$

(23)
$$W_{1\to 6}(z) = \frac{z(1-2z+2z^2)}{(1-z)(1+2z)(1-3z)} = z + 7z^3 + 8z^4 + 51z^5 + 100z^6 + \cdots$$

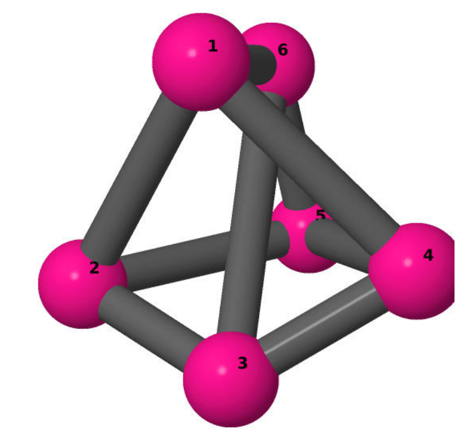

FIGURE 3. The utility graph.

4. UTILITY GRAPH

The adjacency matrix of the Utility Graph of Figure 3 is

$$\begin{pmatrix} 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{pmatrix}. \tag{24}$$

Inverting $1 - zA$ gives [3, A102518,A001019,A013708]

$$W_{1\to 1}(z) = \frac{1-6z^2}{1-9z^2} = 1 + 3z^2 + 27z^4 + 243z^6 + \cdots \tag{25}$$

$$W_{1\to 2}(z) = W_{1\to 4}(z) = W_{1\to 6}(z) = \frac{z}{1-9z^2} = z + 9z^3 + 81z^5 + 729z^6 + \cdots \tag{26}$$

$$W_{1\to 3}(z) = W_{1\to 5}(z) = \frac{3z^2}{1-9z^2} = 3z^2 + 27z^4 + 243z^6 + \cdots \tag{27}$$

5. CUBE

The standard cube (or Y_4-prism) with 8 vertices, 12 edges, and 3 edges at each vertex is labeled as in Figure 4. The adjacency matrix is

$$\begin{pmatrix} 0&1&0&0&0&1&0&1\\ 1&0&1&0&1&0&0&0\\ 0&1&0&1&0&0&0&1\\ 0&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&0\\ 1&0&0&0&1&0&1&0\\ 0&0&0&1&0&1&0&1\\ 1&0&1&0&0&0&1&0 \end{pmatrix}. \tag{28}$$

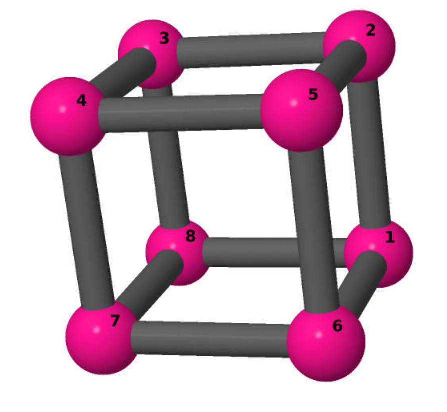

FIGURE 4. Structure of the cubical graph with 8 vertices and 12 edges.

Inverting $1 - zA$ gives [3, A054879,A066443,A125857,A054880]

$$W_{1\to 1}(z) = \frac{1-7z^2}{(1-z^2)(1-9z^2)} = 1 + 3z^2 + 21z^4 + 183z^6 + \cdots , \tag{29}$$

$$W_{1\to 2}(z) = W_{1\to 6}(z) = W_{1\to 8}(z) = \frac{z(1-3z^2)}{(1-z^2)(1-9z^2)} = z + 7z^3 + 61z^5 + 547z^7 + \cdots \tag{30}$$

$$W_{1\to 3}(z) = W_{1\to 5}(z) = W_{1\to 7}(z) = \frac{2z^2}{(1-z^2)(1-9z^2)} = 2z^2 + 20z^4 + 182z^6 + 1640z^8 + \cdots \tag{31}$$

$$W_{1\to 4}(z) = \frac{6z^3}{(1-z^2)(1-9z^2)} = 6z^3 + 60z^5 + 546z^7 + \cdots \tag{32}$$

6. Wagner Graph

The Mobius variant of the cube is the Wagner Graph, Figure 5. The adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{33}$$

Inverting $1 - zA$ gives

$$W_{1\to 1}(z) = \frac{1 - z - 5z^2 + z^3 + 2z^4}{(1-z^2)(1+2z-z^2)(1-3z)} = 1 + 3z^2 + 19z^4 + 10z^5 + 141z^6 + \cdots \tag{34}$$

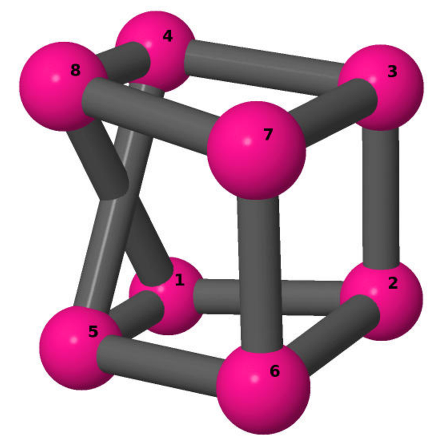

FIGURE 5. Structure of the Wagner graph with 8 vertices and 12 edges.

(35)
$$W_{1\to 2}(z) = W_{1\to 8}(z) = \frac{z(1-2z^2)}{(1+2z-z^2)(1+z)(1-3z)} = z+6z^3+4z^4+45z^5+56z^6+358z^7+\cdots$$

(36)
$$W_{1\to 3}(z) = W_{1\to 7}(z) = \frac{z^2}{(1-z^2)(1-3z)} = z^2+3z^3+10z^4+30z^5+91z^6+273z^6+\cdots, A033113$$

(37)
$$W_{1\to 4}(z) = W_{1\to 6}(z) = \frac{z^2(2+z)}{(1+z)(1-3z)(1+2z-z^2)} = 2z^2+z^3+16z^4+16z^5+126z^6+189z^7+\cdots$$

(38)
$$W_{1\to 5}(z) = \frac{z(1-z-z^2-z^3)}{(1-z^2)(1-3z)(1+2z-z^2)} = z+7z^3+2z^4+51z^5+42z^6+393z^7+\cdots$$

With the auxiliary integer sequence
(39)
$$\frac{1}{1+2z-z^2} = \sum_{n\ge 0} g(n)z^n = 1-2z+5z^2-12z^3+29z^4-70z^5+\cdots; g(n) = -2g(n-1)+g(n-2), A077985$$

we may rewrite these counting sequences of the Wagner Graph as

$$8W_{1\to 1}(n) = 2+3^n+(-1)^n+4g(n)+4g(n-1); \tag{40}$$

$$8W_{1\to 2}(n) = 3^n-(-1)^n+4g(n-1); \tag{41}$$

$$8W_{1\to 4}(n) = 3^n-(-1)^n-4g(n-1); \tag{42}$$

$$8W_{1\to 5}(n) = 3^n+2+(-1)^n-4g(n)-4g(n-1). \tag{43}$$

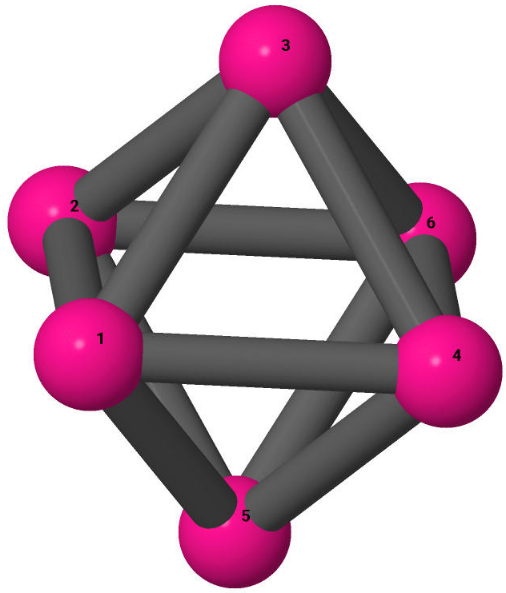

FIGURE 6. The Octahedron.

7. OCTAHEDRON

The adjacency matrix of the Octahedral Graph of Figure 6 is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{44}$$

Inverting $1 - zA$ gives

$$W_{1\to 1}(z) = \frac{1-2z-4z^2}{(1+2z)(1-4z)} = 1 + 4z^2 + 8z^3 + 48z^4 + 160z^5 + \cdots, A054881 \tag{45}$$

(46)
$$W_{1\to 2}(z) = W_{1\to 3}(z) = W_{1\to 4}(z) = W_{1\to 5}(z) = \frac{z}{(1+2z)(1-4z)} = z+2z^2+12z^3+40z^4+176z^5+\cdots, A003683$$

$$W_{1\to 6}(z) = \frac{4z^2}{(1+2z)(1-4z)} = 4zW_{1\to 2}(z) = 4z^2 + 8z^3 + 48z^4 + 160z^5 + \cdots \tag{47}$$

References

1. Cyril Banderier, Christian Krattenthaler, Alan Krinik, Dmitry Kruchinin, Vladimir Kruchinin, David Nguyen, and Michael Wallner, *Explicit formulas for enumeration of lattice paths: Basketball and the kernel method*, Developments in Mathematics, vol. 58, pp. 78–118, 2019. MR 3930453
2. Ernesto Estrada and José A. de la Peña, *Integer sequences from walks in graphs*, Not. Number Theory Disc. Math. **19** (2013), no. 3, 78–84.
3. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2020), https://oeis.org/. MR 3822822
4. C. D. Godsil and B. Mohar, *Walk generating functions and spectral measures of infinite graphs*, J. Lin. Alg. Applic. **104** (1988), 191–206. MR 0960145
5. George Stacey Staples, *A new adjacency matrix for finite graphs*, Adv. Appl. Clifford Alg. **18** (2008), 979–991. MR 2490600

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