OEIS A097344/5

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The similarity between the sequences A097344 and A097345 of the Online Encyclopedia of Integer Sequences is elucidated.

I. DEFINITIONS

A097344 is defined as the numerators of the binomial transform of $1/(n+1)^2$. The sequence of fractions of this binomial transform is by definition

$$f_n \equiv \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)^2} = 1, \frac{5}{4}, \frac{29}{18}, \frac{103}{48}, \frac{887}{300}, \frac{1517}{360}, \ldots; \quad n = 0, 1, 2, 3, \ldots \tag{1}$$

rewritten in terms of a terminating Hypergeometric Function [1, 2]

$$f_n = {}_3F_2\left(\begin{matrix} 1, 1, -n \\ 2, 2 \end{matrix} \mid -1 \right). \tag{2}$$

[The k-sum may be split into even and odd terms,

$$f_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^2} + 2 \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} \frac{1}{(2k+2)^2} \tag{3}$$

and the first, alternating of the two sums written as Bell polynomials with Corollary 2.2 of [3].]

A097345 is defined as the numerators of the partial sums of the binomial transform of $1/(n+1)$. The sequence of fractions of this binomial transform is by definition

$$t_n \equiv \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} = 1, \frac{3}{2}, \frac{7}{3}, \frac{15}{4}, \frac{31}{5}, \frac{21}{2}, \ldots; \quad n = 0, 1, 2, 3, \ldots \tag{4}$$

simplified by [4, 0.155.2] to

$$t_n = \frac{2^{n+1} - 1}{n+1}. \tag{5}$$

t_n generates the partial sums

$$g_n \equiv \sum_{j=0}^{n} t_j = 1, \frac{5}{2}, \frac{29}{6}, \frac{103}{12}, \frac{887}{60}, \frac{1517}{60}, \ldots \quad ; n = 0, 1, 2, 3, \ldots \tag{6}$$

II. CONJECTURE AND NEAR-MISSES

The numerators in (6) appear to coincide with the sequence of numerators of the f_n sequence in (1). Indeed the transition is given by moving (4) into (6)

$$g_n = \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{j}{k} \frac{1}{k+1} \tag{7}$$

where we use a re-summation on the triangular grid of the (k, j) coordinates, $\sum_{j=0}^{n} \sum_{k=0}^{j} = \sum_{k=0}^{n} \sum_{j=k}^{n}$,

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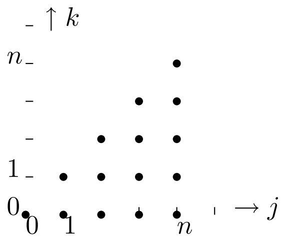
*URL: http://www.strw.leidenuniv.nl/~mathar/progs/a097345.pdf



a resummation $l \equiv j - k$, and then [4, 0.151.1]

$$\begin{aligned} g_n &= \sum_{k=0}^{n}\sum_{j=k}^{n}\binom{j}{k}\frac{1}{k+1} = \sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=k}^{n}\binom{j}{k} = \sum_{k=0}^{n}\frac{1}{k+1}\sum_{l=0}^{n-k}\binom{k+l}{k} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n+1}{k+1} \\ &= \sum_{k=0}^{n}\frac{1}{k+1}\frac{(n+1)!}{(n-k)!(k+1)!} = \sum_{k=0}^{n}\frac{1}{k+1}\cdot\frac{n+1}{k+1}\cdot\frac{n!}{(n-k)!k!} = (n+1)\sum_{k=0}^{n}\frac{1}{(k+1)^2}\cdot\frac{n!}{(n-k)!k!} \\ &= (n+1)\sum_{k=0}^{n}\frac{1}{(k+1)^2}\binom{n}{k} = (n+1)f_n. \end{aligned} \tag{8}$$

To demonstrate that the numerators of g_n and f_n are the same, we have to show that the extra factor $n+1$ in (8) is absorbed by the denominator of f_n, ie, that $n+1$ is a divisor of this denominator. This may actually fail, as embodied by the list of n in A134652 [1].

APPENDIX A: GENERATING FUNCTIONS

The ordinary generating function (o.g.f.) is defined as

$$F(x) \equiv \sum_{n=0}^{\infty} f_n x^n, \tag{A1}$$

denoted by capitalized letters of the associated series of Taylor coefficients.

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}x^n\frac{1}{(k+1)^2} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\binom{n}{k}x^n\frac{1}{(k+1)^2} = \sum_{k=0}^{\infty}\frac{x^k}{(1-x)^{k+1}}\cdot\frac{1}{(k+1)^2} \\ &= \frac{1}{x}\sum_{k=0}^{\infty}\frac{x^{k+1}}{(1-x)^{k+1}}\cdot\frac{1}{(k+1)^2} = \frac{1}{x}\sum_{k=1}^{\infty}\frac{x^k}{(1-x)^k}\cdot\frac{1}{k^2} = \frac{1}{x}Li_2\left(\frac{x}{1-x}\right). \end{aligned} \tag{A2}$$

From (5) we have with [4, 1.513.4]

$$T(x) \equiv \sum_{n=0}^{\infty} t_n x^n = \frac{1}{x}\sum_{n=0}^{\infty}\frac{2^{n+1}-1}{n+1}x^{n+1} = \frac{1}{x}\left[\sum_{n=1}^{\infty}\frac{(2x)^n}{n} - \sum_{n=1}^{\infty}\frac{x^n}{n}\right] = \frac{1}{x}\ln\frac{1-x}{1-2x}. \tag{A3}$$

Since g_n is a polynomial of n multiplied by f_n, the o.g.f. $G(x)$ can be obtained by differentiation of $F(x)$—as also used in the Maple function `GenFpolyMul` in mathar20071126.pdf and in [5]:

$$xF(x) = \sum_{n=0}^{\infty} f_n x^{n+1}; \tag{A4}$$

$$\Rightarrow \frac{d}{dx}(xF) = \sum_{n=0}^{\infty} f_n (n+1) x^n = \sum_{n=0}^{\infty} g_n x^n = G(x). \tag{A5}$$

With [6, 2-1]

$$G(x) = \frac{d}{dx}Li_2\left(\frac{x}{1-x}\right) = \frac{1}{\frac{x}{1-x}}Li_1\left(\frac{x}{1-x}\right)\frac{d}{dx}\frac{x}{1-x} = \frac{1}{x(1-x)}Li_1\left(\frac{x}{1-x}\right). \tag{A6}$$

APPENDIX B: EXTENSIONS

The first differences of f_n are of similar form as the series itself [1]:

$$\begin{aligned} f_{n+1}-f_n &= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{(k+1)^2}-\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)^2}=\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{1}{(k+1)^2}-\sum_{k=1}^{n}\binom{n}{k}\frac{1}{(k+1)^2} \\ &= \frac{1}{(n+2)^2}+\sum_{k=1}^{n}\binom{n+1}{k}\frac{1}{(k+1)^2}-\sum_{k=1}^{n}\binom{n}{k}\frac{1}{(k+1)^2}=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+2)^2}. \end{aligned} \tag{B1}$$

So f_n could be thought of as row $s=1$ of an array $f_{n,s}$ of sequences [1, 2],

$$f_{n,s}\equiv\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+s)^2}=\frac{1}{s^2}\,{}_3F_2\left(\begin{matrix} s,s,-n \\ 1+s,1+s \end{matrix}\mid -1\right);\quad n\ge 0;\quad s\ne 0; \tag{B2}$$

$$f_n\equiv f_{n,1}; \tag{B3}$$

with consecutive rows s related to first differences of previous rows $s-1$:

$$\begin{aligned} f_{n+1,s}-f_{n,s} &= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{(k+s)^2}-\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+s)^2}=\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{1}{(k+s)^2}-\sum_{k=1}^{n}\binom{n}{k}\frac{1}{(k+s)^2} \\ &= \frac{1}{(n+s+1)^2}+\sum_{k=1}^{n}\binom{n+1}{k}\frac{1}{(k+s)^2}-\sum_{k=1}^{n}\binom{n}{k}\frac{1}{(k+s)^2} \\ &= \frac{1}{(n+s+1)^2}+\sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right]\frac{1}{(k+s)^2}-\sum_{k=1}^{n}\binom{n}{k}\frac{1}{(k+s)^2} \\ &= \frac{1}{(n+s+1)^2}+\sum_{k=1}^{n}\binom{n}{k-1}\frac{1}{(k+s)^2}=\frac{1}{(n+s+1)^2}+\sum_{k=0}^{n-1}\binom{n}{k}\frac{1}{(k+s+1)^2}=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+s+1)^2} \\ &= \frac{1}{(1+s)^2}\,{}_3F_2\left(\begin{matrix} 1+s,1+s,-n \\ 2+s,2+s \end{matrix}\mid -1\right)=f_{n,s+1}. \end{aligned} \tag{B4}$$

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