1 Statements and Representatives

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In this paper we consider $\mathbf{R}_n$. Let $e_1, \ldots, e_n$ be the n unit–vectors which generate $\mathbf{R}_n$, then we define by $\ell_n = \{x = \sum a_i e_i | a_1 \geq a_2 \geq \ldots \geq a_n \geq 0\}$. We consider the hypercube H_n which has the vertices $\epsilon_1 e_1 + \ldots + \epsilon_n e_n$. Let $h_1, \ldots, h_{2^n}$ be these vertices of H_n. For each element $x \in \ell_n$ we build 2^n *statements* by taking the innerproduct of x with h_i. We define a statement *true* if $(x, h_i) > 0$, and *false* if $(x, h_i) < 0$. We say that two vectors x and y are *indistinguishable* if all statements produced by x and y are equal. For each set of indistinguishable vectors we chose one vector, which is called the *representative*. There are only a finite number of representatives. In fact this number is bounded by 2^n (=the number of statements, but some combinations of statements do not appear. Therefore the number of representatives is smaller. The question is how many representatives are there?

The rest of the paper does contain the dimensions 3 upto 7. In each section we have 4 tables. In Table 1 we summarize the elements of the hypercube h_i where the vectors are distinguishable. These elements of the hypercube are labeled by $A \ldots Z$ and $a \ldots z$ denoting the opposite elements. Formulas 2 and 3 show all implications of the statements. Finally in Table 4 all representatives are described. The elements of the hypercube denote the statements which are satisfied.

2 Dimension 3

Table 3.1

A	$+--$	a	$-++$

Formula 3.2

$$A$$

Formula 3.3

$$a$$

Table 3.4

1	A	$(1,0,0)$
2	a	$(1,1,1)$

3 Dimension 4

Table 4.1

A	$+---$	a	$-+++$
B	$+--+$	b	$-++-$

Formula 4.2

$$A \Rightarrow B$$

Formula 4.3

$$b \Rightarrow a$$

Table 4.4

1	A	$(1,0,0,0)$
2	Ba	$(2,1,1,1)$
3	b	$(1,1,1,0)$

4 Dimension 5

Table 5.1

A	$+----$	a	$-++++$
B	$+---+$	b	$-+++-$
C	$+--+-$	c	$-++-+$
D	$+-+--$	d	$-+-++$
E	$++---$	e	$--+++$
F	$+--++$	f	$-++--$

Formula 5.2

$$\begin{array}{ccccccccc} & & & & & & f & & \\ & & & & & & \Downarrow & & \\ A & \Rightarrow & B & \Rightarrow & C & \Rightarrow & D & \Rightarrow & E \\ & & & & \Downarrow & & & & \\ & & d & \Rightarrow & F & & & & \end{array}$$

Formula 5.3

$$\begin{array}{ccccccccc} & & & & f & \Rightarrow & D & & \\ & & & & \Downarrow & & & & \\ e & \Rightarrow & d & \Rightarrow & c & \Rightarrow & b & \Rightarrow & a \\ & & \Downarrow & & & & & & \\ & & F & & & & & & \end{array}$$

Table 5.4

1	A	$(1, 0, 0, 0, 0)$
2	Ba	$(3, 1, 1, 1, 1)$
3	Cb	$(2, 1, 1, 1, 0)$
4	DFc	$(3, 2, 2, 1, 1)$
5	Ed	$(2, 2, 1, 1, 1)$
6	e	$(1, 1, 1, 1, 1)$
7	f	$(1, 1, 1, 0, 0)$

5 Dimension 6

Table 6.1

A	$+-----$	a	$-+++++$
B	$+----+$	b	$-++++-$
C	$+---+-$	c	$-+++-+$
D	$+--+--$	d	$-++-++$
E	$+-+---$	e	$-+-+++$
F	$++----$	f	$--++++$
G	$+---++$	g	$-+++--$
H	$+--+-+$	h	$-++-+-$
I	$+-+--+$	i	$-+-++-$
J	$++---+$	j	$--+++-$
K	$+--++-$	k	$-++--+$
L	$+--+++$	l	$-++---$

Formula 6.2

$$
\begin{array}{ccccccccccc}
 & & & & & & & & l & & \\
 & & & & & & & & \Downarrow & & \\
A & \Rightarrow & B & \Rightarrow & C & \Rightarrow & D & \Rightarrow & E & \Rightarrow & F \\
 & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & G & \Rightarrow & H & \Rightarrow & I & \Rightarrow & J \\
 & & & & & & \Downarrow & & \Uparrow & & \\
 & & & & i & \Rightarrow & K & & k & & \\
 & & & & \Downarrow & & \Downarrow & & & & \\
 & & & & e & \Rightarrow & L & & & &
\end{array}
$$

Formula 6.3

$$
\begin{array}{ccccccccccc}
 & & & & l & \Rightarrow & E & & & & \\
 & & & & \Downarrow & & \Downarrow & & & & \\
 & & K & & k & \Rightarrow & I & & & & \\
 & & \Uparrow & & \Downarrow & & & & & & \\
j & \Rightarrow & i & \Rightarrow & h & \Rightarrow & g & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & & & \\
f & \Rightarrow & e & \Rightarrow & d & \Rightarrow & c & \Rightarrow & b & \Rightarrow & a \\
 & & \Downarrow & & & & & & & & \\
 & & L & & & & & & & &
\end{array}
$$

Table 6.4

1	A	$(1,0,0,0,0,0)$
2	Ba	$(4,1,1,1,1,1)$
3	Cb	$(3,1,1,1,1,0)$
4	Dg	$(2,1,1,1,0,0)$
5	DGc	$(5,2,2,2,1,1)$
6	EGd	$(4,2,2,1,1,1)$
7	$EHdg$	$(5,3,3,2,1,1)$
8	EKh	$(3,2,2,1,1,0)$
9	ELk	$(4,3,3,1,1,1)$
0	FGe	$(3,2,1,1,1,1)$
11	$FHeg$	$(4,3,2,2,1,1)$
12	$FIKeh$	$(5,4,3,2,2,1)$
13	Fek	$(3,3,2,1,1,1)$
14	Fi	$(2,2,1,1,1,0)$
15	Gf	$(2,1,1,1,1,1)$
16	Hgf	$(3,2,2,2,1,1)$
17	$IKfh$	$(4,3,3,2,2,1)$
18	$ILfhk$	$(2,2,2,1,1,1)$
19	Jfi	$(3,3,2,2,2,1)$
10	Kj	$(1,1,1,1,1,0)$
20	l	$(1,1,1,0,0,0)$

6 Dimension 7

Table 7.1

A	$+------$	a	$-++++++$
B	$+-----+$	b	$-+++++-$
C	$+----+-$	c	$-++++-+$
D	$+---+--$	d	$-+++-++$
E	$+--+---$	e	$-++-+++$
F	$+-+----$	f	$-+-++++$
G	$++-----$	g	$--+++++$
H	$+----++$	h	$-++++--$
I	$+---+-+$	i	$-+++-+-$
J	$+--+--+$	j	$-++-++-$
K	$+-+---+$	k	$-+-+++-$
L	$++----+$	l	$--++++-$
M	$+---++-$	m	$-+++--+$
N	$+--+-+-$	n	$-++-+-+$
O	$+-+--+-$	o	$-+-++-+$
P	$++---+-$	p	$--+++-+$
Q	$+--++--$	q	$-++--++$
R	$+-+-+--$	r	$-+-+-++$
S	$++--+--$	s	$--++-++$
T	$+-++---$	t	$-+--+++$
U	$++-+---$	u	$--+-+++$
V	$+++----$	v	$---++++$
W	$+---+++$	w	$-+++---$
X	$+--+-++$	x	$-++-+--$
Y	$+-+--++$	y	$-+-++--$
Z	$++---++$	z	$--+++--$
Θ	$+--++-+$	θ	$-++--+-$
Π	$+--+++-$	π	$-++---+$
Σ	$+--++++$	σ	$-++----$

Formula 7.2

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & \sigma & & & & \\
 & & & & & & & & & & \Downarrow & & & & \\
A & \Rightarrow & B & \Rightarrow & C & \Rightarrow & D & \Rightarrow & E & \Rightarrow & F & \Rightarrow & G & & \\
 & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & H & \Rightarrow & I & \Rightarrow & J & \Rightarrow & K & \Rightarrow & L & & \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & & & M & \Rightarrow & N & \Rightarrow & O & \Rightarrow & P & \Rightarrow & Z \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & X & \Leftarrow & W & & Q & \Rightarrow & R & \Rightarrow & S & & \\
 & & & & & & & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & & & o & \Rightarrow & \Theta & & T & \Rightarrow & U & & \\
 & & & & & & \Downarrow & & \Downarrow & & & & \Downarrow & & \\
 & & & & & & k & \Rightarrow & \Pi & & & & V & & \\
 & & & & & & \Downarrow & & \Downarrow & & & & & & \\
 & & & & & & f & \Rightarrow & \Sigma & & & & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
M & \Rightarrow & N & \Rightarrow & O & \Rightarrow & P \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
W & \Rightarrow & X & \Rightarrow & Y & \Rightarrow & Z \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
t & \Rightarrow & r & \Rightarrow & q & &
\end{array}
$$

$$
\begin{array}{ccccccc}
M & \Rightarrow & N & \Rightarrow & O & \Rightarrow & P \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
W & \Rightarrow & X & \Rightarrow & Y & \Rightarrow & Z \\
 & & \Downarrow & & & & \\
Q & \Rightarrow & \Theta & \Rightarrow & \Pi & \Rightarrow & \Sigma
\end{array}
$$

$$
\begin{array}{ccccccccccc}
\sigma & \Rightarrow & \pi & \Rightarrow & \theta & & y & \Rightarrow & x & \Rightarrow & w \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
F & \Rightarrow & K & \Rightarrow & O & & Q & \Rightarrow & R & \Rightarrow & T
\end{array}
$$

Formula 7.3

$$
\begin{array}{ccccccccccccc}
 & & & & \sigma & \Rightarrow & F & & & & & & \\
 & & & & \Downarrow & & \Downarrow & & & & & & \\
v & & & & \pi & \Rightarrow & K & & z & & \sigma & & \\
\Downarrow & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
u & \Rightarrow & t & & \theta & \Rightarrow & O & & y & & \pi & & \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \Downarrow & & \Downarrow & & \\
s & \Rightarrow & r & \Rightarrow & q & & w & \Leftarrow & x & \Leftarrow & \theta & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & & & & & \\
p & \Rightarrow & o & \Rightarrow & n & \Rightarrow & m & & & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & & & & & \\
l & \Rightarrow & k & \Rightarrow & j & \Rightarrow & i & \Rightarrow & h & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & & & \\
g & \Rightarrow & f & \Rightarrow & e & \Rightarrow & d & \Rightarrow & c & \Rightarrow & b & \Rightarrow & a \\
 & & \Downarrow & & & & & & & & & & \\
 & & \Sigma & & & & & & & & & &
\end{array}
$$

Table 7.4

1	A	$(1,0,0,0,0,0,0)$	51	$GQYox$	$(7,6,4,3,3,1,1)$
2	Ba	$(5,1,1,1,1,1,1)$	52	$GRWoq$	$(8,7,4,3,3,2,2)$
3	Cb	$(4,1,1,1,1,1,0)$	53	$GRXoqw$	$(9,8,5,4,3,2,2)$
4	DHc	$(7,2,2,2,2,1,1)$	54	$GTWr$	$(7,6,3,2,2,1,1)$
5	Dh	$(3,1,1,1,1,0,0)$	55	$Gf\pi$	$(4,4,3,1,1,1,1)$
6	EHd	$(6,2,2,2,1,1,1)$	56	$Goqx$	$(5,5,3,2,2,1,1)$
7	$EIdh$	$(8,3,3,3,2,1,1)$	57	Grw	$(4,4,2,2,1,1,1)$
8	EMi	$(5,2,2,2,1,1,0)$	58	Gt	$(3,3,1,1,1,1,1)$
9	EWm	$(7,3,3,3,1,1,1)$	59	Gy	$(2,2,1,1,1,0,0)$
10	Ew	$(2,1,1,1,0,0,0)$	60	Hg	$(3,1,1,1,1,1,1)$
11	FHe	$(5,2,2,1,1,1,1)$	61	Igh	$(5,2,2,2,2,1,1)$
12	$FIeh$	$(7,3,3,2,2,1,1)$	62	$JMgi$	$(7,3,3,3,2,2,1)$
13	$FJMei$	$(9,4,4,3,2,2,1)$	63	$JWgm$	$(4,2,2,2,1,1,1)$
14	$FJew$	$(7,4,4,3,1,1,1)$	64	Jgw	$(5,3,3,3,1,1,1)$
15	FMj	$(4,2,2,1,1,1,0)$	65	$KMgj$	$(6,3,3,2,2,2,1)$
16	$FNWjm$	$(9,5,5,3,2,2,1)$	66	$KNWgjm$	$(7,4,4,3,2,2,1)$
17	$FNjw$	$(5,3,3,2,1,1,0)$	67	$KNgjw$	$(8,5,5,4,2,2,1)$
18	$FQWn$	$(7,4,4,2,2,1,1)$	68	$KQWgn$	$(5,3,3,2,2,1,1)$
19	$FQXnw$	$(8,5,5,3,2,1,1)$	69	$KQXgnw$	$(6,4,4,3,2,1,1)$
20	FQx	$(3,2,2,1,1,0,0)$	70	$KQgx$	$(7,5,5,3,3,1,1)$
21	FWq	$(5,3,3,1,1,1,1)$	71	$KWgq$	$(3,2,2,1,1,1,1)$
22	$FXqw$	$(6,4,4,2,1,1,1)$	72	$KXgqw$	$(4,3,3,2,1,1,1)$
23	$F\Theta qw$	$(7,5,5,2,2,1,1)$	73	$K\Theta gqx$	$(5,4,4,2,2,1,1)$
24	$F\Pi\theta$	$(4,3,3,1,1,1,0)$	74	$K\Pi g\theta$	$(6,5,5,2,2,2,1)$
25	$F\Sigma\pi$	$(5,4,4,1,1,1,1)$	75	$LMgk$	$(5,3,2,2,2,2,1)$
26	GHf	$(4,2,1,1,1,1,1)$	76	$LNWgkm$	$(6,4,3,3,2,2,1)$
27	$GIfh$	$(6,3,2,2,2,1,1)$	77	$LNgkw$	$(7,5,4,4,2,2,1)$
28	$GJMfi$	$(8,4,3,3,2,2,1)$	78	$LOQWgkn$	$(7,5,4,3,3,2,1)$
29	$GJWfm$	$(5,3,2,2,1,1,1)$	79	$LOQXgknw$	$(8,6,5,4,3,2,1)$
30	$GJfw$	$(6,4,3,3,1,1,1)$	80	$LOQgkx$	$(7,5,4,3,3,2,1)$
31	$GKMfj$	$(7,4,3,2,2,2,1)$	81	$LOWgkn(q)$	$(5,4,3,2,2,2,1)$
32	$GKNWfjm$	$(8,5,4,3,2,2,1)$	82	$LOXgkqw$	$(6,5,4,3,2,2,1)$
33	$GKNfjw$	$(9,6,5,4,2,2,1)$	83	$LO\Theta gkqx$	$(7,6,5,3,3,2,1)$
34	$GKXfqw$	$(5,4,3,2,1,1,1)$	84	$LQWgo$	$(4,3,2,2,2,1,1)$
35	$GKQfx$	$(8,6,5,3,3,1,1)$	85	$LQXgow$	$(5,4,3,3,2,1,1)$
36	$GKQWfn$	$(6,4,3,2,2,1,1)$	86	$LQgox$	$(6,5,4,3,3,1,1)$
37	$GKQXfnw$	$(7,5,4,3,2,1,1)$	87	$Lgk\theta$	$(5,5,4,2,2,2,1)$
38	$GKWfq$	$(7,5,4,2,2,2,1)$	88	Lgt	$(2,2,1,1,1,1,1)$
39	$GK\Theta fqx$	$(6,5,4,2,2,1,1)$	89	Ml	$(2,1,1,1,1,1,0)$
40	$GK\Pi f\theta$	$(7,6,5,2,2,2,1)$	90	$NWlm$	$(5,3,3,3,2,2,1)$
41	GMk	$(3,2,1,1,1,1,0)$	91	Nlw	$(3,2,2,2,1,1,0)$
42	$GNWkm$	$(7,5,3,3,2,2,1)$	92	$OQWln$	$(5,4,4,3,3,2,1)$
43	$GNkw$	$(4,3,2,2,1,1,0)$	93	$OQXlnw$	$(7,5,5,4,3,2,1)$
44	$GOQWkn$	$(8,6,4,3,3,2,1)$	94	$OQlx$	$(4,3,3,2,2,1,0)$
45	$GOQXknw$	$(9,7,5,4,3,2,1)$	95	$OWlq$	$(4,3,3,2,2,2,1)$
46	$GOQkx$	$(5,4,3,2,2,1,0)$	96	$OXlqw$	$(5,4,4,3,2,2,1)$
47	$GO\Theta kqx$	$(7,6,5,3,3,2,1)$	97	Olx	$(2,2,2,1,1,1,0)$
48	$GOk\theta$	$(3,3,2,1,1,1,0)$	98	$PQWlo$	$(5,4,3,3,3,2,1)$
49	$GQWo$	$(5,4,2,2,2,1,1)$	99	$PQXlow$	$(6,5,4,4,3,2,1)$
50	$GQXow$	$(6,5,3,3,2,1,1)$	100	Ply	$(3,3,2,2,2,1,0)$

101	QWp	$(3,2,2,2,2,1,1)$
102	$QXpw$	$(4,3,3,3,2,1,1)$
103	$QYpx$	$(5,4,4,3,3,1,1)$
104	$RWpq$	$(5,4,4,3,3,2,2)$
105	$RXpqw$	$(6,5,5,4,3,2,2)$
106	$STWpr$	$(7,6,5,5,4,3,3)$
107	Spt	$(4,4,3,3,3,2,2)$
108	Spw	$(4,3,3,3,2,1,1)$
109	TWs	$(4,3,3,3,2,2,2)$
110	Ust	$(5,5,4,4,3,3,3)$
111	Vu	$(3,3,3,2,2,2,2)$
112	Zpy	$(4,4,3,3,3,1,1)$
113	Π	$(5,1,1,1,1,1,1)$
114	$g\pi$	$(3,3,3,1,1,1,1)$
115	pqx	$(3,3,3,2,2,1,1)$
116	sw	$(2,2,2,2,1,1,1)$
117	v	$(1,1,1,1,1,1,1)$
118	z	$(1,1,1,1,1,0,0)$
119	σ	$(1,1,1,0,0,0,0)$

7 General Dimension

The number of vertices of the hypercube involved in the tables 1 is given by:

n	Number of formulas			
3	1	$2^2 - \binom{3}{1}$	$=$	$4-3$
4	2	$2^3 - \binom{4}{2}$	$=$	$8-6$
5	6	$2^4 - \binom{5}{2}$	$=$	$16-10$
6	12	$2^5 - \binom{6}{3}$	$=$	$32-20$
7	29	$2^6 - \binom{7}{3}$	$=$	$64-35$
8	58	$2^7 - \binom{8}{4}$	$=$	$128-70$
9	130	$2^8 - \binom{9}{4}$	$=$	$256-126$
10	260	$2^9 - \binom{10}{5}$	$=$	$512-252$
11	562	$2^{10} - \binom{11}{5}$	$=$	$1024-462$
12	1124	$2^{11} - \binom{12}{6}$	$=$	$2048-924$
13	2380	$2^{12} - \binom{13}{6}$	$=$	$4096-1716$
14	4760	$2^{13} - \binom{14}{7}$	$=$	$8192-3432$
15	9949	$2^{14} - \binom{15}{7}$	$=$	$16384-6435$

The formula for the number of vertices of the hypercube involved is:

$$F_n = 2\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 2} \binom{n-1}{k} + \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1} = 2^{n-1} - \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$