A181605 is a subsequence of A092340

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After verifying that 7 is in A092340, it suffices to prove that if p is a prime $\equiv 7 \bmod 10$ and $q = p + 2$ is a prime, then p and q both divide $F_{p^2} + F_{2p}$, where F_n are the Fibonacci numbers.

Since p is not a quadratic residue mod 5 while q is, by quadratic reciprocity 5 is not a quadratic residue mod p but is a quadratic residue mod q. We will use the Binet formula

$$F_n = \frac{\phi_+^n - \phi_-^n}{\sqrt{5}}$$

where $\phi_\pm = (1 \pm \sqrt{5})/2$ are roots of the polynomial $x^2 - x - 1$. This will work in $GF(q)$, the integers mod q, where ϕ_+ and ϕ_- are roots of $x^2 - x - 1$ in that field, and showing that $F_{p^2} + F_{2p} = 0$ in that field will show that the integer $F_{p^2} + F_{2p}$ is divisible by q. On the other hand, for p we will go to the quadratic extension $GF(p^2)$ where 5 has a square root and $\phi_\pm$ are roots of $x^2 - x - 1$. Showing that $F_{p^2} + F_{2q} = 0$ in $GF(p^2)$ will show that the integer $F_{p^2} + F_{2p}$ is divisible by p.

In $GF(p^2)$, since this has characteristic p,

$$\phi_+^p + \phi_-^p = (\phi_+ + \phi_-)^p = 1^p = 1 \text{ and } \phi_+^p \phi_-^p = (-1)^p = -1$$

so ϕ_+^p and ϕ_-^p are also roots of $x^2 - x - 1$. But there are only two of those roots, namely ϕ_+ and ϕ_-. Now ϕ_+^p can't be ϕ_+ (because only the p elements of $GF(p)$ are roots of the polynomial $X^p - X$), so it must be ϕ_-, and ϕ_-^p must be ϕ_+. Moreover $\phi_+^{p^2} = \phi_-^p = \phi_+$ and similarly $\phi_-^{p^2} = \phi_-$. So

$$F_{p^2} + F_{2p} = \frac{\phi_+^{p^2} - \phi_-^{p^2}}{\sqrt{5}} + \frac{\phi_+^{2p} - \phi_-^{2p}}{\sqrt{5}} = \frac{\phi_+ - \phi_-}{\sqrt{5}} + \frac{\phi_-^2 - \phi_+^2}{\sqrt{5}} = F_1 - F_2 = 0$$

In $GF(q)$, $\phi_+^q = \phi_+$ and $\phi_-^q = \phi_-$, $\phi_+^p = \phi_+^{q-2} = \phi_+^{-1} = -\phi_-$ and similarly $\phi_-^p = -\phi_+$. Moreover $\phi_+^{p^2} = (-\phi_-)^p = -\phi_+$ and similarly $\phi_-^{p^2} = -\phi_1$. So

$$F_{p^2} + F_{2p} = \frac{\phi_+^{p^2} - \phi_-^{p^2}}{\sqrt{5}} + \frac{\phi_+^{2p} - \phi_-^{2p}}{\sqrt{5}} = \frac{-\phi_+ + \phi_-}{\sqrt{5}} + \frac{\phi_+^2 - \phi_-^2}{\sqrt{5}} = -F_1 + F_2 = 0$$