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Fibonacci sequences with relative prime initial conditions, and the binary quadratic form $[1,\ 1,\ -1]$

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Abstract

This is a comment on *Brousseau*'s [3] ordering of *Fibonacci* sequences with relative prime inputs. It relates to the integer proper solutions of the indefinite binary quadratic form $x^2 + x\,y - y^2 = N$ with integer N. This is the principal reduced form of discriminant 5, a member of a 2−cycle of forms. The corresponding representative parallel primitive forms lead to the solutions, expressible in terms of *Chebyshev*'s polynomials $\{S(n,3)\}_{n\,\geq\,-2}$.

1 The binary quadratic form $[1,\ 1,\ -1]$

For details on indefinite binary quadratic forms see the *Buell* [2] and *Scholz-Schoeneberg* [9] references, and also the author's paper [6], given as a link in A225953, where proofs are given for some of the later given statements. See also the authors paper [7] (with a link in A324251) on *Pell* cycles and graphs. Computations have been done with the help of Maple [4].

The primitive binary quadratic form $x^2 + x\,y - y^2$ representing an integer N is abbreviated as $[a,\ b,\ c] = [1,\ 1,\ -1]$, and has discriminant $b^2 - 4\,a\,c = 5 =$ A003658(2). The class number h(5) of this from is $1 =$ A087048(0). This means that there is only one cycle of reduced forms, the one related to the principal reduced form, *viz* $F_p = [1,\ 1,\ -1]$ (see [6], *Lemma 2*). The right-neighbor form $F' = [-1,\ 1,\ 1]$ is the other member of this 2−cycle. The proper equivalence transformation from F_p to F' is obtained by the determinant +1 matrix $\mathbf{R}(t) = Matrix([[0,-1],[1,t]])$,with $t = -1$, and F' transforms back to F_p with $\mathbf{R}(1)$. Hence the matrix of the automorphic equivalence transformation for this cycle (automorphic matrix, for short) is $\mathbf{Auto}(5) = \mathbf{R}(-1)\,\mathbf{R}(1)$, which is

$$\mathbf{Auto}(5) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \tag{1}$$

The k-th power of this matrix is (*via* the *Cayley-Hamilton* theorem)

$$\mathbf{Auto}(5)^k = \begin{pmatrix} S(k-1,\,3) - S(k-2,\,3)\,, & S(k-1,\,3) \\ S(k-1,\,3) & ,\; S(k,\,3) - S(k-1,\,3 \end{pmatrix}, \quad \text{for } k \in \mathbb{Z}\,, \tag{2}$$

where the *Chebyshev* polynomial system $\{S(n,3)\}_{n\,\in\,\mathbb{Z}}$ enters, with $S(n,3) =$ A001906$(n+1)$, for $n = -1,\,0,\,\dots$, with $S(-2,\,3) = -1$, and $S(-|k|,\,3) = -S(|k| - 2,\,3)$, for $k \in \mathbb{N}_0$.

$\mathbf{Auto}(5)^k$, for $k \in \mathbb{Z} \setminus 0$, gives all integer proper solutions $(x(N,j;k),\,y(N,j;k))^\top$ ($\top$ for transposed, and proper meaning that $\gcd(x(N,j;k),\,y(N,j;k)) = 1$) of $x^2 + x\,y - y^2 = N$, originating from each of the $j = 1,\,\dots,\,\#pfsols(N)$ proper nonnegative fundamental solutions (*pfsols*) $(x(N,j;0),\,y(N,j;0))^\top$ by

$$\begin{pmatrix} x(N,j;k) \\ y(N,j;k) \end{pmatrix} = \mathbf{Auto}(5)^k \begin{pmatrix} x(N,j;0) \\ y(N,j;0) \end{pmatrix}, \quad \text{for } j = 1,\,2,\,\dots,\,\#pfsols(N)\,, \quad k \in \mathbb{Z}\,. \tag{3}$$

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In general there is no formula for giving these *pfsols*, only prescriptions scanning certain candidates are known. (*e.g.*, for the general *Pell* equations there are the inequalities given by Nagell [5], *Theorems* 108 and 108b).
We prefer to use the method which searches for all so-called *representative parallel primitive forms* (*rpapfs*) for a given discriminant *Disc* (here $Disc = 5$) and a representation of N. (This has been explained in detail, with references, for the general *Pell* equations in [7]).

A remark on the primitive but not reduced form $\widehat{F} = [1, -1, -1]$ of discriminant 5. It is properly equivalent to the principal form $F_p = [1, 1, -1]$ by two $R-$transformations, *viz* $\mathbf{R}(1)\,\mathbf{R}(0)$. This means (up to an overall sign change) that each relative prime (proper) solution (X, Y) representing $\widehat{F}$ with value N is a proper solution (x, y) of F_p representing N with $x = X - Y$ and $y = Y$. *Vice versa*, (X, Y), with $X = x + y$ and $Y = y$, solves $\widehat{F} = N$ properly if (x, y) solves $F_p = N$ properly. This fact will be used later in connection with the approach of *Brother Alfred* (*A. Brousseau*) [3]. The exchange of indeterminates in F_p leads to $F_p(y\,x) = -\widehat{F}(x, y)$.

The principal form F_p is symmetric under $x \to -x$ together with $y \to -y$, hence, by an overall sign change, we may restrict to solutions of $F_p(x, y) = N$ with $x > 0$. Whenever a given formula produces a non-positive solution for x, such an overall sign change is applied but not always mentioned.

Because the change $x \to -y$ and $y \to x$ produces from a solution with positive N a solution for $-N$ (possibly with the mentioned overall sign change to obtain again $x > 0$) we restrict ourselves to positive N in the following. There is no solution for $N = 0$, because this change for $N = 0$ implies $(x, y) = (-y, x)$, but $\gcd(0, 0) \neq 1$.

It is clear that proper solutions (x, y) can lead only to odd N, because the non-trivial case odd-odd leads to a result congruent to $1\,(mod\,2)$.

Each possible value of N represented properly by F_p is congruent to 0 or -1 or $+1$ modulo 5, as can be seen by checking the 15 cases $x \geq y$ modulo 5. However, not all $0, -1, +1\,(mod\,5)$ numbers appear, *e.g.*, $N = 5^2 = 25$ has no proper solution. There are conjectures on the set $\mathcal{N}$ of allowed N values given in comments on A089270. This paper is intended to prove them.
At this stage we can state the following *Lemma* which will be needed in the discussion of *section 3*. Part **iii)** appeared as a conjecture for solutions of $x^2 - x - 1 = 0\,(mod\,N)$, and can be proven similarly (see *section 3*).

Lemma 1:

i) *Every pair* $(x, y = 1)$, *with* $x \in \mathbb{N}$, *is a positive fundamental proper solution of* $F_p = N \geq 1$, *where* $N = N(x) = x^2 + x - 1 = 2\,T(x) - 1 =$ *A028387*$(x-1)$, *with the triangular numbers* $T =$ *A000217*, *is odd, and satisfies* $N(x) \equiv 0$ *or* 1 *or* $4\,(mod\,5)$.
ii) *The sequence* $\{N(x)\}_{x\geq 1}$ *is a proper sub-sequence of A089270 that records* $\mathcal{N}$, *the set of values of* N *for all possible proper solutions of* $x^2 + x\,y - y^2 = N$. *See also Table 4, and* N *in Table 8 with* $k = 1$.
iii) *The indefinite binary quadratic form* $x^2 + x\,y - y^2$ *has a proper representation of* $N \geq 1$ *if and only if* N *solves the congruence* $x^2 + x - 1 \equiv 0\,(mod\,N)$.

Proof:
i) This special case is obvious. Note that in the case $F_p = N = 1$ on can replace the positive fundamental solution $(1, 1)$ by the non-negative one $(1, 0)$, and $(1, 1)^\top = \mathbf{Auto(5)}\,,(1, 0)^\top$.
ii) Also obvious.
iii) This follows from the representative parallel primitive forms (*rpapfs*) for discriminant *Disc* (here Disc $=5$) and representation $N \geq 1$. The class number is $1 =$ A087048(0), which tells that there is only one cycle of reduced primitive forms, and all *rpapfs* reduce to the principal form of this cycle. See [9], paragraph 27, pp. 104-105 (where only primitive forms are considered). For the procedure for the odd *Disc* case see [7], p. 4.
These *rpapfs* are $[N, b(j), c(j)]$ with $j \in \{0, 1, ..., N-1\}$ such that $c(j) = \frac{j\,(j+1) - (5-1)/4)}{N}$ becomes an integer number, and $b(j) = 2\,j + 1$. But this means to solve the congruence $j^2 + j - 1 \equiv 0\,(mod\,N)$ for j.

The possible values for N will be found in *section 3*, and the trivial solution $(1, 0)$ of the *rpapf*s will be important. $\square$

The formula eq. (2) leads to the following result on the k-family of each of the $j \in \{1, 2, ..., \#pfsols(N)\}$ solutions for a given non-negative fundamental proper solution $x(N, j; 0) = a > 0$ and $y(N, j; 0) = b \geq 0$ with $a - b > 0$, implying $N > 0$.

$$\begin{aligned} x(N, j; k) &= (a + b)\, S(k-1, 3) - a\, S(k-2, 3), \\ y(N, j; k) &= b\, S(k, 3) + (a - b)\, S(k-1, 3), \quad \text{for } k \in \mathbb{Z}\,. \end{aligned} \tag{4}$$

$x(N, j; 1) \geq y(N, j; 1)$, and equality holds only for $N = 1$, and $x(N, j; k) > y(N, j; k)$ for $k \geq 2$. For negative k it is clear from the rule for negative indices of the S polynomials that $y(N, j; -|k|) < 0$ for $|k| \in \mathbb{N}$, hence $x(N, j; -|k|) > y(N, j; -|k|)$ trivially, because by an overall sign change always $x(N, j; -|k|) > 0$. These statements follow by using the $S-$ recurrence and the positivity $S(k, 3)$ for $k \geq 0$. *E.g.*, for $k \geq 0$ one finds $x(N, j; -|k|) - y(N, j; -|k|) = -((a - b)\, S(k-2, 3) + b\, S(k-1, 3))$, which is $-b \leq 0$ for $k = 1$, and equality holds only for $N = 1$.

In *Table 1* the *rpapf*s are listed for representable $N = N(n) =$ A089270(n) values, for $n \geq 1$. One obtains from each of these $\#pfsols(N)$ parallel forms a corresponding *pfsol* by repeated $\mathbf{R}(t)$ transformations until one reaches the principal form F_p. The corresponding t-tuples are listed as $\vec{t}(N, j) = (t_1(N, j), \ldots t_{tmax(N,j)}(N, j))$. It is known (see *e.g.*, [9], Satz 79, p. 113) that every primitive (!) form is equivalent to a reduced form (in our case to F_p).

$$\begin{pmatrix} x(N, j; 0) \\ y(N, j; 0) \end{pmatrix} = \mathbf{R}^{-1}(t_{tmax(N,j)}(N, j)) \cdots \mathbf{R}^{-1}(t_1(N, j)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } j = 1, 2, ..., \#pfsols(N)\,. \tag{5}$$

is a fundamental solution, but not necessarily a positive one. However application of positive powers of the $\mathbf{Auto}(5)$ matrix will lead to a first positive fundamental form.

Example: $N = 11 =$ A089270(3), with $\#pfsols(11) = 2$ *rpapf*s $Pa1 = [11, 7, 1]$ and $Pa2 = [11, 15, 5]$ with $\vec{t}(11, 1) = (4)$, and $\vec{t}(11, 2) = (1, -2)$, with the intermediate form $[5, -5, 1]$. Thus

$$\begin{pmatrix} x(11, 1; 0) \\ y(11, 1; 0) \end{pmatrix} = \mathbf{R}^{-1}(4) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \tag{6}$$

In this case the first positive *pfsol* for $F_p = 11$ is

$$\begin{pmatrix} x(11, 1; 1) \\ y(11, 1; 1) \end{pmatrix} = \mathbf{Auto}(5) \begin{pmatrix} x(11, 1; 0) \\ y(11, 1; 0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \tag{7}$$

Similarly, the second parallel form leads first to a solutions with x and y negative, so an overall sign flip gives the first positive *pfsol*:

$$-\begin{pmatrix} x(11, 2; 0) \\ y(11, 2; 0) \end{pmatrix} = -\mathbf{R}^{-1}(-2)\, \mathbf{R}^{-1}(1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \tag{8}$$

The first family (also known as class) of solutions is then, with eq. (2),

$$\begin{pmatrix} S(k-1, 3) - S(k-2, 3)\,, & S(k-1, 3) \\ S(k-1, 3) & , \ S(k, 3) - S(k-1, 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \text{for } k \in \mathbb{Z}\,. \tag{9}$$

This leads to $x(11, 1; k) =$ A013655$(2\,k) = [3, 5, 12, 31, 81, 212, 555, 1453, ...]$, and $y(11, 1; k) =$ A013655$(2\,k + 1) = [2, 7, 19, 50, 131, 343, 898, 2351, ...]$, for $k \in \mathbb{N}_0$. For the second family based on the *ppfsol* (positive proper fundamental solution) $(3, 1)^\top$ one finds the bisection of A104449: $x(11, 2; k) =$ A104449$(2\,k)$ and $y(11, 2; k) =$ A104449$(2\,k + 1)$. These are the two Fibonacci sequences with the relative prime initial conditions $(a(0), b(0)) = (3, 2)$ and $(3, 1)$, respectively.

From eq. (2) one finds for any given positive proper fundamental solution $(a(N,j),\, b(N,j))^\top$ the ordinary generating functions (*o.g.f.*s) for the solutions $(x(N,j;k),\, y(N,j;k))$ for $j \in \{1, 2, ..., \#pfsols\}$ and $k \in N_0$, named $(Gx(N,j;t),\, Gy(N,j;t))$, the results

$$Gx(N,j;t) \;=\; \frac{a(N,j)\,-\,t\,(2\,a(N,j)\,-\,b(N,j))}{1\,-\,3\,t\,+\,t^2},\quad Gy(N,j;t) \;=\; \frac{b(N,j)\,+\,t\,(a(N,j)\,-\,b(N,j))}{1\,-\,3\,t\,+\,t^2}\,. \tag{10}$$

These *o.g.f.*s coincide, for each given N and j, with the bisection of *Fibonacci* sequences $\{F(a,b;n)\}_{n\geq 0}$ with inputs $F(a,b;0) = a$ and $F(a,b;1) = b$, and $F(a,b;n) = F(a,b;n-1) + F(a,b;n-2)$, for $n \geq 2$, *viz*

$$GF_{even}(a,b;t) \;:=\; \sum_{k=0}^{\infty} F(a,b;2\,k)\,t^k \;=\; \frac{a\,-\,t\,(2\,a\,-\,b)}{1\,-\,3\,t\,+\,t^2}, \tag{11}$$

$$GF_{odd}(a,b;t) \;:=\; \sum_{k=0}^{\infty} F(a,b;2\,k+1)\,t^k \;=\; \frac{b\,+\,t\,(a\,-\,b)}{1\,-\,3\,t\,+\,t^2}\,. \tag{12}$$

2 Ordering of Fibonacci sequences with relative prime initial conditions

Brother Alfred [3] considered *Fibonacci* sequences with relative prime nonnegative initial conditions $(a,\, b)$, with $a < b$ and defined for these sequences a certain order.

We give a slightly different approach here. In effect all pairs of relative prime non-negative integers $(n,\, k)$ with $n \geq k \geq 1$ and $\gcd(n,k) = 1$ are first collected in a certain way in an array with row label n and row length $\varphi(n)$ =A000010(n). The array of positive integers relative prime to n is A038566, for $n >= 1$, which we name $AGCD$ with entry $AGCD(n,\, m)$, $m = 1, 2, ..., \varphi(n)$. But instead of A038566(1, 1) $= 1$ we take $AGCD(1,\,1) := 0$. Then in the array $A(n,\, m)$, for $n \geq 1$, of pairs of relative prime integers one takes for all n with $\varphi(n) = 1$ or 2 the pair $(n, AGCD(n,\, m))$, for $m = 1, 2, ..., \varphi(n)$. Hence the first four rows are $[(1,0)]$, $[(2,1)]$, $[(3,1),\,(3,2)]$, $[(4,1),\,(4,3)]$, and row $n = 6$ is $[(6,1),\,(6,5)]$. For row n with $\varphi(n) = 2\,l$, with $l \geq 2$ one considers a reordered row of A038566(n), namely $Sequence(AGCD(n,\,k), AGCD(n,\,\varphi(n)-(k-1)))$ for $k = 1, 2, ..., \varphi(n)/2$. Thus the reordered row $n = 5$ of AGCD(n,k) becomes 1, 4, 2, 3. This reordered array $AGCD$ is named $AGCD'$. Finally, row n of an array AP of pairs is obtained by $AP(n,\,m) = (n, AGCD'(n,\,m))$, for $n \geq 1$. This array AP is shown in *Table 2*. For the corresponding representable values N for $F_p(x = n,\, y = AGCD'(n,\,m)$ see *Table 3*.

In *Table 2* two neighboring pairs appear within square brackets, for $n \geq 3$. The number of these pairs of pairs is named $\#N(n)$. The reason for this grouping of two pairs is that $(x1, y1)$ and $(x2, y2)$ are positive fundamental solutions leading to the same $N = x^2 + x\,y - y^2$ value. See *Table 3*. That pairs collected in a square bracket lead to the same N value is trivial, because (n,k) and $(n,n-k)$ satisfy both $N = n^2 + n\,k - k^2$, but only pairs with $\gcd(n,k) = 1$ qualify (implying that also $\gcd(n,n-k) = 1$ because $\gcd(n,-k) = \gcd(n,k)$).

$\#N(n)$ tells how many representable numbers N are obtained from pairs of array AP with first entry n. (For $n = 1$ and $n = 2$ there is no pair, and $\#N(n)$ is set to 1). *E.g.*, $\#N(7) = 3$, because $N = 55,\, 59,\, 61$ have solutions with first entry 7. Of course, some representable N numbers may have solutions from pairs belonging to different n values; *e.g.*, $N = 209 = 11 \cdot 19$ has four proper fundamental solutions (this N being the first instance with two square bracket pairs), namely from one bracket pair for $n = 13$ and one from $n = 14$: $[(13,5),\,(13,8)]$ and $[(14,1),\,(14,13)]$ (see *Table* 4).

In *Table 4* the square bracket terms from array AP of *Table 2* which lead to the same representable value N are collected. Thus for $N = 1$ we used the proper fundamental solution $(1,0)$, not the positive one $(1,1)$, obtained by applying **Auto**(5).

An ordering of *Fibonacci* sequences $\{F(a,b,n\}_{n\geq 0}$ with relative prime initial conditions $F(a,b;0) = a$ and $F(a,b;1) = b$, with $a > b$, is now obtained by using $a = x$ and $b = y$ of the proper fundamental solutions, ordering with rising N, and within like N the pairs (x,y) are ordered lexicographically. *E.g.*, the first sequence is $(1,0)$, the fourth is $(3,2)$, and $(14,1)$ with $N = 209$ is the 57th sequence (in *Brother Alfred*'s counting it corresponds to his $(1,15)$ appearing at position 55.

This order is similar to the one of *Brother Alfred* with value $D = N$, but his pairs (f_0, f_1) satisfy $D = f_1^2 - f_1 f_0 - f_0^2$. Thus, using the remark on the form $\widehat{F} = []1, -1, -1]$ in *section 1* above, we have to take $x = f_1 - f_0$ and $y = f_0$, or $f_0 = y$ and $f_1 = x + y$. Thus the ordinary *Fibonacci* sequence $F =$ A00045, *Brother Alfred*'s $(0,1)$ becomes now $(1,0)$ which is $F(1,0;n) =$ A00045$(n-1)$, for $n \geq 1$, and $F(1,0;0) = 1 = F(-1)$. Similarly, *Brother Alfred*'s second sequence $(1,3)$ is shifted *Lucas* A000032$(n+1)$, for $n \geq 0$. This becomes $F(2,1;n) =$ A000032(n) directly. Also his fourth sequence $(1,4)$ is directly A000285, whereas our $F(3,1;n) =$ A104449$(n) =$ A000285$(n-1)$, for $n \geq 0$. Note a typo in [3] for $D = 31$; the second pair should be $(3,8)$ (not $(2,8)$.

In conclusion the two orderings of *Fibonacci* sequences with relative prime initial conditions differ. *Brother Alfred* used rising non-negative pairs of integers whereas we use falling ones. The translation between these pairs has been given. Because $x + y = f_1$ and $y = f_0$ our sequences start with an extra x for $n = 0$ and then coincide with *Brother Alfred*'s sequence with a shifted index by $+1$. In the case of more then two pairs for one value $D = N$, like for $N = 209$, the order of the pairs, hence of the sequences, differs.

3 Diophantine equations $\mathbf{x^2 + x - 1 \equiv 0\,(mod\,N)}$, and $\mathbf{x^2 - 5 \equiv 0\,(mod\,N)}$

In this section it will be seen that the Diophantine equations $x^2 + x - 1 \equiv 0\,(mod\,N)$, and $x^2 - 5 \equiv 0\,(mod\,N)$, with positive N have only solutions for $N = 1$, and for N with a prime number factorization with only odd primes congruent to $\pm 1\,(mod\,5)$ and their powers, multiplied by either 1 or 5. The primes $\pm 1\,(mod\,5)$ are given on A038872 (without the leading 5). From *Lemma 1* part **iii**) these numbers then coincide with $N = N(n) =$ A089270(n). This structure of these N has been conjectured by *T. D. Noe* as a comment in A089270 from Nov 14 2010. For the congruence $x^2 - x - 1 \equiv 0\,(mod\,N)$ this has also been conjectured by *T. D. Noe* in a Nov 04 2009 comment.

In the following we give a proof for the structure of these N values and the number of representative solutions, starting by listing 7 ingredients for this proof. An additional point **8)** is added to show how solutions with composite N are computed from the ones for $N = 5$ and powers of primes congruent to $\pm 1\,(mod\,5)$.

1) The solutions of the two quadratic congruences

$$x^2 + x - 1 \equiv 0\,(mod\,N), \quad \text{and} \quad X^2 - X - 1 \equiv 0\,(mod\,N), \quad \text{for} \quad N \in \mathbb{N}, \tag{13}$$

are related by $X = x + 1$. This is trivial. See also the more general remark on the relation between the solutions of the forms $F_p = [1, 1, -1] = N$ and $\widehat{F} = [1, -1, -1] = N$ in *section 1*. Therefore, only the first case will be considered in the following.

2) The congruence $x^2 - 5 \equiv 0\,(mod\,N)$ means that 5 is a quadratic residue modulo N, in short $5RN$. The case $N = 1$ is trivial because every integer number is a solution, and the representative solution (in the first non-negative residue class) is 0. For $x^2 + x - 1 \equiv 0\,(mod\,1)$ the representative solution is also 0.

The case $N = 5$ is special because $\gcd(5, N) = 5 \neq 1$. There is just one representative solution from the first non-negative residue class, *viz* 0. For the other congruence the only representative solution for $N = 5$ is 2.

3) The relation between solutions of $x^2 + x - 1 \equiv 0\,(mod\,N)$ and those of $\tilde{x}^2 - 5 \equiv 0\,(mod\,4\,N)$, is given by $2\,x_{\pm} + 1 = \pm\,\tilde{x}$, where $\tilde{x} = \sqrt{5 + 4\,N\,k}$, with the least positive integer number k producing a positive $\tilde{x}$ (later these minimal k values for representable N and the (odd) $\tilde{x}$ are listed in *Table 8*). This is then evaluated modulo N. Hence for the solutions $x_{\pm}$ one needs the solutions $\tilde{x}\,(mod\,N)$ (it seems that $\tilde{x} < N$ already).
For $N = 1$ one finds from the square root formula $\tilde{x} = 3$ with $k = 1$, and $x_+ = 1$, $x_- = -2$, evaluating modulo 1 to the representatives $\tilde{x} = 0$ and $x_+ = x_- = 0$, which is clear immediately, because every integer satisfies the congruences modulo 1.
For $N = 5$ the results are $\tilde{x} = 5$ for $k = 1$ and $x_+ = 2$, $x_- = -3$. Evaluated modulo 5 with the representatives $\tilde{x} = 0$ and $x_+ = x_- = 2$.

For all other solvable positive $N \neq 1, 5$ the representative solutions come in pairs. See *Tables* 6 and 7 for these representative solutions for solvable $N = N(n)$ (to be determined later) for $n = 1, 2, ..., 60$. The pairs of solutions are given within brackets. In *Table 6* these pairs sum to $N - 1$, and in *Table 7* they add to N. Composite N values are underlined.
It follows from *section 1* that the first congruence, as a special case of $F_p = N$, for $N \geq 1$, with $y = 1$, can have only solutions with odd N from the sets 0, ± 1 modulo 5 (but as will be shown only the proper subset of solvable $N = N(n)$ values stated in the preamble of this section will survive).

4) For $N \equiv 0\,(mod\,5)$ the prime number factorization of N has only 5 as a factor; no higher powers of 5 are allowed. For $\tilde{x}^2 - 5 \equiv 0\,(mod\,N)$, with $N = 5^{e_5}\,\widehat{N}$, where $e_5 \in \mathbb{N}$, and 5 does not divide $\widehat{N}$, it follows that $5\,(1 + 5^{e_5 - 1}\,m\,\widehat{N}) = \tilde{x}^2$, with an integer number m, but $1 + 5^{e_5-1}\,m\,\widehat{N} \equiv 1\,(mod\,5)$, for $e_5 - 1 \geq 1$. Hence a second factor 5 for $\tilde{x}^2 \neq 0$ is missing, and only $e_5 = 1$ remains. If $\tilde{x}^2 = 0$ then $m\,\widehat{N} = -1$ and $e_5 = 1$ also. For general solvable N the exponent of prime number 5 is then $e_5 \in \{0, 1\}$.

5) Each integer solution of $\tilde{x}^2 - 5 \equiv 0\,(mod\,N)$, for positive odd $N > 1$, is also a solution of all congruences $\tilde{x}^2 - 5 \equiv 0\,(mod\,p_i^{e_i})$, with the powers of odd primes p_i, with $e_i \in \mathbb{N}$, appearing the prime number factorization of N (the prime 5 has exponent 0 or 1). Moreover, the number of representative solutions of the congruence modulo N is given by the product of the number of such solutions for these prime power moduli. (See *e.g.*, [1], Theorem 5.28, p. 118-119.)
Each of these two congruences with moduli of prime powers $p_i^{e_i}$, with $e_i \geq 2$, is reduced to congruences with only single p_is, and in the case at hand with the integer polynomial $f(x) = x^2 - 5$ the lifting theorem given in [1], Theorem 5.30, pp. 121-122, needs only part (a), with $f'(r) = 2\,r \not\equiv 0\,(mod\,p_i)$. In this case there is a unique lifting from each solution r of $x^2 - 5 \equiv 0\,(mod\,p_i^{e_i - 1})$, for $e_i \geq 2$, to a solution a of $x^2 - 5 \equiv 0\,(mod\,p_i^{e_i})$, and $a = r + q\,p_i^{e_i - 1}$ with the unique solution q of the linear congruence $q\,2\,r + k \equiv 0\,(mod\,p_i)$, where k is given by $f(r) = r^2 - 5 = k\,p_i^{e_i - 1}$.
The indirect proof that $2\,r \not\equiv 0\,(mod\,p_i)$, *i.e.*, $r \neq m\,p_i$, for each solution $r^2 - 5 = k\,p_i^{e_i - 1}$ is trivial because then $5 = p_i\,(m^2\,p_i - k\,p_i^{e_i - 2})$, a contradiction because $p_i \neq 5$.

6) First one studies the congruence $x^2 - 5 \equiv 0\,(mod\,p)$, for each odd prime $p \neq 5$. With this prime modulus and degree 2 it has either no solution or 2 (different) representative solutions. The *Legendre* symbol $\left(\frac{5}{p}\right)$ equals $+1$ iff $5Rp$, -1 iff not$(5Rp)$. The computation of the symbol uses first *Gauss*'s quadratic reciprocity law (see *e.g.*, [1] Theorem 9.8, p. 185) and then *Euler*'s criterion ([1], Theorem 9.6, p. 182).

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) \equiv p^2\,(mod\,5) = \begin{cases} +1, & \text{for } p \equiv \pm 1\,(mod\,5), \\ \\ -1, & \text{for } p \equiv \pm 2\,(mod\,5). \end{cases} \quad . \tag{14}$$

One uses (for $p \neq 5$) $1/\left(\frac{p}{5}\right) = \left(\frac{p}{5}\right)$. The four cases for p congruent to 5 lead to the last equation. (See also [10], p. 159.)
Thus, for positive $N \not\equiv 0\,(mod\,5)$ a solvable N has only powers of (odd) primes 1 or $4\,(mod\,5)$ in its decomposition. Elements of these two classes of primes will be denoted by $p_{+1,i}$ or $p_{-1,j}$, with exponents $e_{+1,i}$ or $e_{-1,j}$ respectively.

If $N \equiv 0\,(mod\,5)$ the factor 5 (no higher power of it) is also present from statement **4)**.

7) Lemma 2:

The number of representative solutions $\#Sol(N)$ for each of the two congruences modulo $N = 5^{e_5} \prod_{i=1}^{r_{+1}} p_{+1,i}^{e_{+1,i}} \prod_{j=1}^{r_{-1}} p_{-1,j}^{e_{-1,j}}$ is thus given by

$$\#Sol(N) = 2^{r_{+1}+r_{-1}}. \tag{15}$$

Proof: See the first part of statement **5)**. $\#Sol(N)$ for $x^2 - 5 \equiv 0\,(mod\,N)$ is identical with $\#Sol(N)$ for $x^2 + x - 1 \equiv 0\,(mod\,N)$ due to statement **3)**. For the first congruence the unique sequential lifting (see statement **5)**) from solutions modulo a prime $p \neq 5$ to solutions of powers of this prime implies that the number of solutions stays 0 or 2. If 5 divides N then statements **2)** and **4)** show that the unique representative solution for $p = 5$ does not matter for this counting of solutions. □

This ends also the proof of the statement given at the beginning of this section which is given here as

Proposition:

i) *The congruences $x^2 + x - 1 \equiv 0\,(mod\,N)$, $x^2 - x - 1 \equiv 0\,(mod\,N)$, and $x^2 - 5 \equiv 0\,(mod\,N)$ have for positive N proper solutions precisely for*

$$N = 1, \text{ and } N = 5^{e_5} \prod_{i=1}^{r_{+1}} p_{+1,i}^{e_{+1,i}} \prod_{j=1}^{r_{-1}} p_{-1,j}^{e_{-1,j}}, \tag{16}$$

where $p_{+1,i}$ and $p_{-1,j}$ are primes congruent to $+1$ and -1 modulo 5, respectively, and $e_5 \in \{0, 1\}$, $e_{+,i}$, $e_{-1,j} \in \mathbb{N}_0$.

ii) *he number of representative solutions $\#Sol(N)$ is given in Lemma 2.*

iii) *These numbers N coincide by Lemma 1, part* **iii)**, *with the ones from A089270 which collects all representable positive numbers N for the principal form $F_p = [1, 1 - 1]$ of discriminant 5.*

8) In order to find the solutions of these congruences (it is sufficient to know the ones for $x^2 - 5 \equiv 0\,(mod\,N)$) one needs for composite N with products of 5 and powers of primes $\pm 1\,(mod\,5)$ the Chinese remainder theorem *(CRT)* (*e.g.*, [1],Theorem 5.26, p. 117-118). For the solutions of powers of a prime from the ones for the prime by the lifting theorem (point **5)**) one has to solve linear congruences. We give two examples.

a) $N = 11^2$. The two representative solutions for the congruence $x^2 - 5 \equiv 0\,(mod\,11)$ are found by finding the smallest positive k such that $\tilde{x} = \sqrt{5 + k \cdot 11}$ is integer, viz $k = 1$. Hence the two solutions are $\pm\tilde{x} = \pm 4\,(mod\,11)$, and in the first non-negative residue system modulo 11 this is $\tilde{x} = 4$ and 7 (see *Table 7*, $n = 3$). For the lifting to 11^2 one has to solve (see point **5)**) for each of these two solutions r the linear congruence $q(r)\,2\,r + k(r) \equiv 0\,(mod\,11)$. Here $k(4) = 1$ and $k(7) = 4$. Then $q(4) = 4$ (from $8 \cdot 4 + 1 = 33$) and $q(7) = 6$ (from $14 \cdot 6 + 4 = 88$). Therefore the two solutions for 11^2 are $a(4) = 4 + 4 \cdot 11 = 48$ and $a(7) = 7 + 6 \cdot 11 = 73$. See *Table 7*, row $n = 17$. In the next lifting step one finds for $N = 11^3 = 1331$ the two representative solutions 73 and 1258.

The two solutions for $x^2 + x - 1 \equiv 0\,(mod\,11^2)$ are with point **3)** obtained directly from $\pm\tilde{x}$, with $\tilde{x} = \sqrt{5 + k \cdot 4 \cdot 11^2}$, and $k = 11$ leads to $\tilde{x} = 73$. Hence $x_\pm = \frac{\pm\tilde{x}-1}{2}\,(mod\,121)$ or representatives $x_+ = 36$, and $x_- = 121 - 74/2 = 84$ (see *Table 6*, row $n = 17$).

Note that the odd solution 73 obtained from the lifting to 11^2 satisfies $73^2 - 5 = 44 \cdot 121$, therefore it appeared again in the solution modulo $4 \cdot 11^2$, and $-73\,(mod\,484) = 411$. From $410/2 = 205 \equiv 84\,(mod\,121)$, the other solution x_- is found directly above.

b) $N = 11^2 \cdot 19 = 2299$. The $2 \cdot 2 = 4$ representative solutions r_i, for $i = 1, 2, ..., 4$, are found *via CRT* from the pairs of representative solutions for 11^2 and 19, *viz* (48, 9), (48, 10), (73, 9) and (73, 10). The first pair leads to $r_1 = 48 + k \cdot 121 = 9 + l \cdot 19$, hence $r_1 = 1016$, for $k = 8$ and $l = 53$. Also $r_2 = 48$, $r_3 = 2299 - 1016 = 1283$ and $r_4 = 2299 - 48 = 2251$.

We close by explaining *Table 8*. It is obvious that each $(x,\, y\, =\, 1)$, for $x \in \mathbb{N}$ is one of the proper fundamental solution of $F_p = N$, for some $N = N(x) \in \mathbb{N}$ (for $N = 1$ one takes, as explained above, $(1,\, 1)$ instead of $(1,\, 0)$ from *Table 4*). However, not every positive N which has proper solutions of $F_p = N$, *viz* $N = N(n) =$ A089127(n), for $n \in \mathbb{N}$, has a fundamental solution $(x,\, 1)$. *E.g.*, $N(6) = 31$ has one pair of fundamental solutions $(5, 2)$ and $(5,\, 3)$ (see *Table 4*).
The numbers N which have fundamental solutions $(x, 1)$ are given in A028387$(x-1)$, for $x \in \mathbb{N}$. See *Lemma 1* **i)**.
For numbers N which have $k \neq 1$ in *Table 8* one wants to determine a number $\widehat{N} = k\,N$, with a positive integer k such that $F_p = \widehat{N}$ has a solution $(\hat{x},\, 1)$. This leads to the equation $2\,\hat{x}_{\pm} + 1 = \pm\tilde{x}$, with $\tilde{x} = \sqrt{5 + 4\,k\,N}$, and the least positive k which produces a positive integer $\tilde{x}$ is searched. These k have to have the structure given in the *Proposition* for N, because they satisfy $x^2 - 5 = 4\,k\,N \equiv 0\,(mod\, k)$ with the known structure for k instead of N. These k are found for $n = 1,\, 2,\, ...,\, 60$, including the $N(n)$ values which have fundamental solution $(x(n),\, 1)$ (for $k = k(n) = 1$). If one could prove that for each $N(n) =$ A089270(n) without a fundamental solution $(x, 1)$ there exits a $k(n)$ such that $\widehat{N}(n) = k(n)\,N(n)$ has a fundamental solution $(\hat{x},\, 1)$, then one would have another, independent proof of the *Proposition*, because then, $N(n) = \frac{\widehat{N}(n)}{k(n)}$. Therefore, the structure of N would follow from the ones of $\widehat{N}$ and k, just by subtracting from the exponents of the primes $\pm 1\,(mod\, 5)$ or 5 of $\widehat{N}(n)$ the ones of the smaller $k(n)$.
This seems not to be obvious. $\widehat{N}$, having solution $(\hat{x},\, 1)$, is guaranteed to have the structure one wants to prove for N, and not all values appear, *viz* precisely the ones with $k = 1$ are missing. The *Pell* equation $u^2 - 5\,v^2 = M$ should have solutions $(u,\, 1)$ for $M = 4\,k\,N(n)$ for some $k = k(N) \geq 1$ and all $n \in \mathbb{N}$ (which is now proved as a corollary to the *Proposition*, but not independently). In *Table 8* the column $\tilde{x} = \tilde{x}(n)$ solves $\tilde{x}(n)^2 - 5 = 4\,k\,N(n)$, for k given in the third column, hence $u = u(n) = \tilde{x}(n)$. See also *Table 7* where these $\tilde{x}$ values appear together with $N - \tilde{x}$.
Note that $\widehat{N}$ may occur for different N values. *E.g.*, $89 \cdot N(5) = 79 \cdot N(57) = 5 \cdot 79 \cdot 89 = 35155$.

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Table 1: **Representative parallel primitive forms (rpapfs) for discriminant 5 and N with their t-tuples**

n	**N**	**rpapfs**	**t**-tuple	**(x(N,j;0), y(N,j;0))**
1	**1**	$[1,1,-1] = F_p$	no $\vec{t}$	$(1,0)$
2	**5**	$[5,5,1]$	(3)	$(3,-1)$
3	**11**	$[11,7,1],\ [11,15,5]$	$(4),(1,-2)$	$(4,-1),\ (-3,-1)$
4	**19**	$[19,9,1],\ [19,29,11]$	$(5),\ (1,-3)$	$(5,-1),\ (-4,-1)$
5	**29**	$[29,11,1],\ [29,47,19]$	$(6),\ (1,-4)$	$(6,-1),\ (-5,-1)$
6	**31**	$[31,25,5],\ [31,37,11]$	$(2,-2),\ (1,-2)$	$(-5,-2),\ (5,3)$
7	**41**	$[41,13,1],\ [41,69,29]$	$(7),\ (1,-5)$	$(7,-1),\ (-6,-1)$
8	**55**	$[55,15,1],\ [55,95,41]$	$(8),\ (1,-6)$	$(8,-1),\ (-7,-1)$
9	**59**	$[59,51,11],\ [59,67,19]$	$(2,-3),\ (1,-2,-2,-2)$	$(-7,-2),\ (-7,-5)$
10	**61**	$[61,35,5],\ [61,87,31]$	$(3,-2),\ (1,-3,-2)$	$(-7,-3),\ (7,4)$
11	**71**	$[71,17,1],\ [71,125,55]$	$(9),\ (1,-7)$	$(8,-1),\ (-8,-1)$
12	**79**	$[79,59,11],\ [79,99,31]$	$(2,-2,-2),\ (1,-2,-3)$	$(8,5),\ (8,3)$
13	**89**	$[89,19,1],\ [89,159,71]$	$(10),\ (1,-8)$	$(10,-1),\ (-9,-1)$
14	**95**	$[95,85,19],\ [95,105,29]$	$(2,-4),\ (1,-2,-2,-2,-2)$	$(-9,-2),\ (9,7)$
15	**101**	$[101,45,5],\ [101,157,61]$	$(4,-2),\ (1,-4,-2)$	$(-9,-4),\ (9,5)$
16	**109**	$[109,21,1],\ [109,197,89]$	$(11),\ (1,-9)$	$(11,-1),\ (-10,-1)$
17	**121**	$[121,73,11],\ [121,169,59]$	$(3,-3),\ (1,-2,-2,-2)$	$(-10,-3),\ (-10,-7)$
18	**131**	$[131,23,1],\ [131,239,109]$	$(12),\ (1,-10)$	$(12,-1),\ (-11,-1)$
19	**139**	$[139,127,29],\ [139,151,41],$	$(2,-5),\ (1,-2,-2,-2,-2,-2)$	$(-11,-2),\ (-11,-9)$
20	**145**	$[145,105,19],\ [145,185,59]$	$(2,-2,-2,-2),\ (1,-2,-4)$	$(-11,-8),\ (11,3)$
21	**149**	$[149,81,11],\ [149,217,79]$	$(3,-2,-2),\ (1,-3,-3)$	$(11,7),\ (11,4)$
22	**151**	$[151,55,5],\ [151,247,101]$	$(5,-2),\ (1,-5,-2)$	$(-11,-5),\ (11.6)$
23	**155**	$[155,25,1],\ [155,285,131]$	$(13),\ (1,-11)$	$(13,-1),\ (-12,-1)$
24	**179**	$[179,149,31],\ [179,209,61]$	$(2,-3,-2),\ (1,-2,-2,-3)$	$(12,7),\ (-12,-5)$
25	**181**	$[181,27,1],\ [181,335,155]$	$(14),\ (1,-12)$	$(14,-1),\ (-13,-1)$
26	**191**	$[191,177,41],\ [191,205,55]$	$(2,-6),\ (1,-2,-2,-2,-2,-2,-2)$	$(-13,-2),\ (13,11)$
27	**199**	$[199,123,19],\ [199,275,95]$	$(3,-4),\ (1,-3,-2,-2,-2)$	$(-13,-3),\ (13,10)$
28	**205**	$[205,95,11],\ [205,315,121]$	$(4,-3),\ (1,-4,-2,-2)$	$(-13.-4),\ (-13.-9)$
29	**209**	$[209,29,1],\ [209,161,31],$ $[209,257,79],\ [209,389,181]$	$(15),\ (2,-2,-3),\ (1,-2,-3,-2)\,(1,-13)$	$(15,-1),\ (13,5),\ (-13,-8),\ (-14,-1)$
30	**211**	$[211,65,5],\ [211,357,151]$	$(6,-2),\ (1,-6,-2)$	$(-13,-6),\ (13,7)$

Table 2: AP array of pairs of relatively prime nonnegative integers

n	**pairs of relative prime pairs**	**#N(n)**
1	[(1, 0)]	1
2	[(2, 1)]	1
3	[(3, 1), (3, 2)]	1
4	[(4, 1), (4, 3)]	1
5	[(5, 1], (5, 4)], [(5, 2), (5, 3)]	2
6	[(6, 1), (6, 5)]	1
7	[(7, 1), (7, 6)], [(7, 2), (7, 5)], [(7, 3), (7, 4)]	3
8	[(8, 1), (8, 7)], [(8, 3), (8, 5)]	2
9	[(9, 1), (9, 8)], [(9, 2), (9, 7)], [(9, 4), (9, 5)]	3
10	[(10, 1), (10, 9)], [(10, 3), (10, 7)]	2
11	[(11, 1), (11, 10)], [(11, 2), (11, 9)], [(11, 3), (11, 8)], [(11, 4), (11, 7)], [(11, 5), (11, 6)]	5
12	[(12, 1), (12, 11)], [(12, 5), (12, 7)]	2
13	[(13, 1), (13, 12)], [(13, 2), (13, 11)], [(13, 3), (13, 10)], [(13, 4), (13, 9)], [(13, 5), (13, 8)], [(13, 6), (13, 7)]	6
14	[(14, 1), (14, 13)], [(14, 3), (14, 11)], [(14, 5), (14, 9)]	3
15	[(15, 1), (15, 14)], [(15, 2), (15, 13)], [(15, 4), (15, 11)], [(15, 7), (15, 8)]	4
16	[(16, 1), (16, 15)], [(16, 3), (16, 13)], [(16, 5), (16, 11)], [(16, 7), (16, 9)]	4
17	[(17, 1), (17, 16)], [(17, 2), (17, 15)], [(17, 3), (17, 14)], [(17, 4), (17, 13)], [(17, 5], (17, 12)], [(17, 6), (17, 11)], [(17, 7), (17, 10)], [(17, 8), (17, 9)]	8
18	[(18, 1), (18, 17)], [(18, 5], (18, 13)], [(18, 7), (18, 11)]	3
19	[(19, 1), (19, 18)], [(19, 2], (19, 17)], [(19, 3), (19, 16)], [(19, 4), (19, 15)], [(19, 5], (19, 14)], [(19, 6), (19, 13)], [(19, 7), (19, 12)], [(19, 8), (19, 11)], [(19, 9), (19, 10)]	9
20	[(20, 1), (20, 19)], [(20, 3), (20, 17)], [(20, 7), (20, 13)], [(20, 9], (20, 11)]	4
...	**...**	**...**

Table 3: Array of representable numbers N of F_p for arrayAP

n	**N**	**#N(n)**
1	[1]	1
2	[5]	1
3	[11, 11]	1
4	[19, 19]	1
5	[29, 29], [31, 31]	2
6	[41, 41]	1
7	[55, 55], [59, 59], [61, 61]	3
8	[71, 71], [79, 79]	2
9	[89, 89], [95, 95], [101, 101]	3
10	[109, 109], [121, 121]	2
11	[131, 131], [139, 139], [145, 145], [149, 149], [151, 151]	5
12	[155, 155], [179, 179]	2
13	[181, 181], [191, 191], [199, 199], [205, 205], [209, 209], [211, 211]	6
14	[209, 209], [229, 229,], [241, 241]	3
15	[239, 239], [251, 251], [269, 269], [281, 281]	4
16	[271, 271], [295, 295], [311, 311], [319, 319]	4
17	[305, 305], [319, 319], [331, 331], [341, 341], [349, 349], [355, 355], [359, 359], [361, 361]	8
18	[341, 341], [389, 389], [401, 401]	3
19	[379, 379], [395, 395], [409, 409], [421, 421], [431, 431], [439, 439], [445, 445], [449, 449], [451, 451]	9
20	[419, 419], [451, 451], [491, 491], [499, 499]	4
...	**...**	**...**

Table 4
Array of proper fundamental solutions (pfsols) representing N with form $\mathbf{F_p} = [\mathbf{1}, \mathbf{1}, -\mathbf{1}]$

n	N	pfsols $(\mathbf{x}, \mathbf{y})$	n	N	pfsols $(\mathbf{x}, \mathbf{y})$
1	1	$[(1,0)]$	31	229	$[(14,3),(14,11)]$
2	5	$[(2,1)]$	32	239	$[(15,1),(15,14)]$
3	11	$[(3,1),(3,2)]$	33	241	$[(14,5),(14,9)]$
4	19	$[(4,1),(4,3)]$	34	251	$[(15,2),(15,13)]$
5	29	$[(5,1],(5,4)]$	35	269	$[(15,4),(15,11)]$
6	31	$[(5,2),(5,3)]$	36	271	$[(16,1),(16,15)]$
7	41	$[(6,1),(6,5)]$	37	281	$[(15,7),(15,8)]$
8	55	$[(7,1),(7,6)]$	38	295	$[(16,3),(16,13)]$
9	59	$[(7,2),(7,5)]$	39	305	$[(17,1),(17,16)]$
10	61	$[(7,3),(7,4)]$	40	311	$[(16,5),(16,11)]$
11	71	$[(8,1),(8,7)]$	41	319	$[(16,7),(16,9)],[(17,2),(17,15)]$
12	79	$[(8,3),(8,5)]$	42	331	$[(17,3),(17,14)]$
13	89	$[(9,1),(9,8)]$	43	341	$[(17,4),(17,13)],[(18,1),(18,17)]$
14	95	$[(9,2),(9,7)]$	44	349	$[(17,5),(17,12)]$
15	101	$[(9,4),(9,5)]$	45	355	$[(17,6),(17,11)]$
16	109	$[(10,1),(10,9)]$	46	359	$[(17,7),(17,10)]$
17	121	$[(10,3),(10,7)]$	47	361	$[(17,8),(17,9)]$
18	131	$[(11,1),(11,10)]$	48	379	$[(19,1),(19,18)]$
19	139	$[(11,2),(11,9)]$	49	389	$[(18,5),(18,13)]$
20	145	$[(11,3),(11,8)]$	50	395	$[(19,2),(19,17)]$
21	149	$[(11,4),(11,7)]$	51	401	$[(18,7),(18,11)]$
22	151	$[(11,5),(11,6)]$	52	409	$[(19,3),(19,16)]$
23	155	$[(12,1),(12,11)]$	53	419	$[(20,1),(20,19)]$
24	179	$[(12,5),(12,7)]$	54	421	$[(19,4),(19,15)]$
25	181	$[(13,1),(13,12)]$	55	431	$[(19,5),(19,14)]$
26	191	$[(13,2),(13,11)]$	56	439	$[(19,6),(19,13)]$
27	199	$[(13,3),(13,10)]$	57	445	$[(19,7),(19,12)]$
28	205	$[(13,4),(13,9)]$	58	449	$[(19,8),(19,11)],$
29	209	$[(13,5),(13,8)],[(14,1),(14,13)]$	59	451	$[(19,9),(19,10)],[(20,3),(20,17)]$
30	211	$[(13,6),(13,7)]$	60	461	$[(21,1),(21,20)]$

Table 5
A-numbers for Fibonacci sequences with relative prime inputs for n = 1..30

n	N	inputs (a, b) = (x, y)	A-numbers
1	**1**	$(1, 0)$	A00045$(n-1)$
2	**5**	$(2, 1)$	A00032(n)
3	**11**	$(3, 1)$, $(3, 2)$	A104449(n), A013655(n)
4	**19**	$(4, 1)$, $(4, 3)$	A022095$(n-1)$, A022120$(n-1)$
5	**29**	$(5, 1)$, $(5, 4)$	A022096$(n-1)$, A022130$(n-1)$
6	**31**	$(5, 2)$, $(5, 3)$	A022113$(n-1)$, A022121$(n-1)$
7	**41**	$(6, 1)$, $(6, 5)$	A022097$(n-1)$, A022136$(n-1)$
8	**55**	$(7, 1)$, $(7, 6)$	A022098$(n-1)$, A022388$(n-1)$
9	**59**	$(7, 2)$, $(7, 5)$	A022114$(n-1)$, A022137$(n-1)$
10	**61**	$(7, 3)$, $(7, 4)$	A022122$(n-1)$, A022131$(n-1)$
11	**71**	$(8, 1)$, $(8, 7)$	A022099$(n-1)$,A022389$(n-1)$
12	**79**	$(8, 3)$, $(8, 5)$	A022123$(n-1)$, A022138$(n-1)$
13	**89**	$(9, 1)$, $(9, 8)$	A022100$(n-1)$, A022390$(n-1)$
14	**95**	$(9, 2)$, $(9, 7)$	A022115$(n-1)$, A190995(n)
15	**101**	$(9, 4)$, $(9, 5)$	A022132$(n-1)$, A022139$(n-1)$
16	**109**	$(10, 1)$, $(10, 9)$	A022101$(n-1)$, A184959(n)
17	**121**	$(10, 3)$, $(10, 7)$	A022124$(n-1)$, A190996(n)
18	**131**	$(11, 1)$, $(11, 10)$	A022102$(n-1)$, A185691$(n-1)$
19	**139**	$(11, 2)$, $(11, 9)$	A022116$(n-1)$, A206422$(n+1)$
20	**145**	$(11, 3)$, $(11, 8)$	A022125$(n-1)$, A206420$(n+1)$
21	**149**	$(11, 4)$, $(11, 7)$	A022133$(n-1)$, A206419$(n+1)$
22	**151**	$(11, 5)$, $(11, 6)$	A022140$(n-1)$, A166025$(n-1)$
23	**155**	$(12, 1)$, $(12, 11)$	A022103$(n-1)$, A097657$(n-1)$
24	**179**	$(12, 5)$, $(12, 7)$	A022141$(n-1)$,A206423$(n+1)$
25	**181**	$(13, 1)$, $(13, 12)$	A022104$(n-1)$, A186620$(n-1)$
26	**191**	$(13, 2)$, $(13, 11)$	A022117$(n-1)$, A206607$(n+1)$
27	**199**	$(13, 3)$, $(13, 10)$	A022126$(n-1)$, A206608$(n+1)$
28	**205**	$(13, 4)$, $(13, 9)$	A022134$(n-1)$, A206609$(n+1)$
29	**209**	$(13, 5)$, $(13, 8)$	A022142$(n-1)$, A206610$(n+1)$
		$(14, 1)$, $(14, 13)$	A022105$(n-1)$, A206564$(n+1)$
30	**211**	$(13, 6)$, $(13, 7)$	A206612$(n+1)$, A206611$(n+1)$

Table 6

Representative solutions of $\mathbf{x^2 + x - 1 \equiv 0 \,(\mathrm{mod}\, N(n))}$, for $\mathbf{n = 1, 2, ..., 60}$

n	N	solutions (mod N)	n	N	solutions (mod N)
1	**1**	0	**31**	**229**	$(81, 147)$
2	**5**	2	**32**	**239**	$(15, 223)$
3	**11**	$(3, 7)$	**33**	**241**	$(51, 189)$
4	**19**	$(4, 14)$	**34**	**251**	$(117, 133)$
5	**29**	$(5, 23)$	**35**	**269**	$(71, 197)$
6	**31**	$(12, 18)$	**36**	**271**	$(16, 254)$
7	**41**	$(6, 34)$	**37**	**281**	$(37, 243)$
8	<u>**55**</u>	$(7, 47)$	**38**	<u>**295**</u>	$(92, 202)$
9	**59**	$(25, 33)$	**39**	<u>**305**</u>	$(17, 287)$
10	**61**	$(17, 43)$	**40**	**311**	$(58, 252)$
11	**71**	$(8, 62)$	**41**	<u>**319**</u>	$(139, 179)$, $(150, 168)$
12	**79**	$(29, 49)$	**42**	**331**	$(116, 214)$
13	**89**	$(9, 79)$	**43**	<u>**341**</u>	$(18, 322)$, $(80, 260)$
14	<u>**95**</u>	$(42, 52)$	**44**	**349**	$(143, 205)$
15	**101**	$(22, 78)$	**45**	<u>**355**</u>	$(62, 292)$
16	**109**	$(10, 98)$	**46**	**359**	$(105, 253)$
17	<u>**121**</u>	$(36, 84)$	**47**	<u>**361**</u>	$(42, 318)$
18	**131**	$(11, 119)$	**48**	**379**	$(19, 359)$
19	**139**	$(63, 75)$	**49**	**389**	$(151, 237)$
20	<u>**145**</u>	$(52, 93)$	**50**	<u>**395**</u>	$(187, 207)$
21	**149**	$(40, 108)$	**51**	**401**	$(111, 289)$
22	**151**	$(27, 123)$	**52**	**409**	$(129, 279)$
23	<u>**155**</u>	$(12, 142)$	**53**	**419**	$(20, 398)$
24	**179**	$(74, 104)$	**54**	**421**	$(110, 310)$
25	**181**	$(13, 167)$	**55**	**431**	$(90, 340)$
26	**191**	$(88, 102)$	**56**	**439**	$(69, 369)$
27	**199**	$(61, 137)$	**57**	<u>**445**</u>	$(187, 257)$
28	<u>**205**</u>	$(47, 157)$	**58**	**449**	$(165, 283)$
29	<u>**209**</u>	$(14, 194)$, $(80, 128)$	**59**	<u>**451**</u>	$(47, 403)$, $(157, 293)$
30	**211**	$(32, 178)$	**60**	**461**	$(21, 439)$

Table 7
Representative solutions of $x^2 - 5 \equiv 0 \pmod{N(n)}$, for n = 1, 2, ..., 60

n	N	solutions (mod N)	n	N	solutions (mod N)
1	**1**	0	**31**	**229**	(66, 163)
2	**5**	0	**32**	**239**	(31, 208)
3	**11**	(4, 7)	**33**	**241**	(103, 138)
4	**19**	(9, 10)	**34**	**251**	(16, 235)
5	**29**	(11, 18)	**35**	**269**	(126, 143)
6	**31**	(6, 25)	**36**	**271**	(33, 238)
7	**41**	(13, 28)	**37**	**281**	(75, 206)
8	**55**	(15, 40)	**38**	**295**	(110, 185)
9	**59**	(8, 51)	**39**	**305**	(35, 270)
10	**61**	(26, 35)	**40**	**311**	(117, 194)
11	**71**	(17, 54)	**41**	**319**	(18, 301), (40, 279)
12	**79**	(20, 59)	**42**	**331**	(98, 233)
13	**89**	(19, 70)	**43**	**341**	(37, 304), (161, 180)
14	**95**	(10, 85)	**44**	**349**	(62, 287)
15	**101**	(45, 56)	**45**	**355**	(125, 230)
16	**109**	(21, 88)	**46**	**359**	(148, 211)
17	**121**	(48, 73)	**47**	**361**	(85, 276)
18	**131**	(23, 108)	**48**	**379**	(39, 340)
19	**139**	(12, 127)	**49**	**389**	(86, 303)
20	**145**	(40, 105)	**50**	**395**	(20, 375)
21	**149**	(68, 81)	**51**	**401**	(178, 223)
22	**151**	(55, 96)	**52**	**409**	(150, 259)
23	**155**	(25, 130)	**53**	**419**	(41, 378)
24	**179**	(30, 149)	**54**	**421**	(200, 221)
25	**181**	(27, 154)	**55**	**431**	(181, 250)
26	**191**	(14, 177)	**56**	**439**	(139, 300)
27	**199**	(76, 123)	**57**	**445**	(70, 375)
28	**205**	(95, 110)	**58**	**449**	(118, 331)
29	**209**	(29, 180), (48, 161)	**59**	**451**	(95, 356), (136, 315)
30	**211**	(65, 146)	**60**	**461**	(43, 418)

Table 8

Finding $\hat{N}(n) = k(n) \cdot N(n)$ for pfsols $(\hat{x}(n), 1)$ of $F_p = [1, 1, -1]$ for $n = 1, 2, ..., 60$

n	N	k	$\tilde{x}$	$\hat{x}$	$\widehat{N}$	n	N	k	$\tilde{x}$	$\hat{x}$	$\widehat{N}$
1	**1**	1	3	1	1	**31**	**229**	29	163	81	6641
2	**5**	1	5	2	5	**32**	**239**	1	31	15	239
3	**11**	1	7	3	11	**33**	**241**	11	103	51	2651
4	**19**	1	9	4	19	**251**	**251**	55	235	117	13805
5	**29**	1	11	5	29	**35**	**269**	19	143	71	5111
6	**31**	5	25	12	155	**36**	**271**	1	33	16	271
7	**41**	1	13	6	41	**37**	**281**	5	75	37	1405
8	**55**	1	15	7	55	**38**	**295**	29	185	92	8555
9	**59**	11	51	25	649	**39**	**305**	1	35	17	305
10	**61**	5	35	17	305	**40**	**311**	11	117	58	3421
11	**71**	1	17	8	71	**41**	**319**	61	279	139	19459
12	**79**	11	59	29	869	**42**	**331**	41	233	116	13571
13	**89**	1	19	9	89	**43**	**341**	1	37	18	341
14	**95**	19	85	42	1805	**44**	**349**	59	287	143	20591
15	**101**	5	45	22	505	**45**	**355**	11	125	62	3905
16	**109**	1	21	10	109	**46**	**359**	31	211	105	11129
17	**121**	11	73	36	1331	**47**	**361**	5	85	42	1805
18	**131**	1	23	11	131	**48**	**379**	1	39	19	379
19	**139**	29	127	63	4031	**49**	**389**	59	303	151	22951
20	**145**	19	105	52	2755	**50**	**395**	89	375	187	35155
21	**149**	11	81	40	1639	**51**	**401**	31	223	111	12431
22	**151**	5	55	27	755	**52**	**409**	41	259	129	16769
23	**155**	1	25	12	155	**53**	**419**	1	41	20	419
24	**179**	31	149	74	5549	**54**	**421**	29	221	110	12209
25	**181**	1	27	13	181	**55**	**431**	19	181	90	8189
26	**191**	41	177	88	7831	**56**	**439**	11	139	69	4829
27	**199**	19	123	61	3791	**57**	**445**	79	375	187	35155
28	**205**	11	95	47	2255	**58**	**449**	61	331	165	27389
29	**209**	1	29	14	209	**59**	**451**	5	95	47	2255
30	**211**	5	65	32	1055	**60**	**461**	1	43	21	461