Conjecture of A081832

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Proposition 1. *$a(n+1) - a(n) \in \{0, 1\}$ for all $n \geq 1$ where $a(n) = a(n+1-2 \cdot a(n-1)) + a(n-2 \cdot a(n-2))$, with $a(1) = a(2) = 1$.*

Proof. Let us assume that $a(k) - a(k-1) \in \{0, 1\}$ for all $2 \leq k \leq n$. We know that this is correct for small n and we proceed by induction. We should show that $a(n+1) - a(n) \in \{0, 1\}$ for all possible cases. By definition equations are below.

$$a(n+1) = a(n+2-2 \cdot a(n)) + a(n+1-2 \cdot a(n-1)) \tag{1.1}$$
$$a(n) = a(n+1-2 \cdot a(n-1)) + a(n-2 \cdot a(n-2)) \tag{1.2}$$
$$a(n-1) = a(n-2 \cdot a(n-2)) + a(n-1-2 \cdot a(n-3)) \tag{1.3}$$
$$a(n-2) = a(n-1-2 \cdot a(n-3)) + a(n-2-2 \cdot a(n-4)) \tag{1.4}$$

From 1.1 and 1.2, $a(n+1) - a(n) = a(n+2-2 \cdot a(n)) - a(n-2 \cdot a(n-2))$.

Case 1. $a(n) = a(n-1) + 1$ and $a(n-1) = a(n-2)$. At this case,

$$\begin{aligned} a(n+1) - a(n) &= a(n+2-2 \cdot a(n)) - a(n-2 \cdot a(n-2)) \\ &= a(n+2-2 \cdot (a(n-1)+1)) - a(n-2 \cdot a(n-1)) \\ &= a(n-2 \cdot a(n-1)) - a(n-2 \cdot a(n-1)) \\ &= 0. \end{aligned}$$

In this case, there is more than above. From 1.2 and 1.3, $a(n) - a(n-1) = a(n+1-2 \cdot a(n-1)) - a(n-1-2 \cdot a(n-3)) = 1$. Since our initial assumption exists about slowness, this can be only possible with $n+1-2 \cdot a(n-1) > n-1-2 \cdot a(n-3)$ and $a(n-1) - a(n-3) < 1$, that is, $a(n-1) = a(n-2) = a(n-3)$. Since we know that $a(n+1) - a(n) = 0$ in above, this case guarantees that existence of two consecutive 1 in first differences is impossible. In other words, if $a(n) - a(n-1) = 1$, then $a(n-1) = a(n-2)$ and $a(n) = a(n+1)$ are only options and this means that we have $2^2 - 1 = 3$ cases in order to complete this proof.

Case 2. $a(n) = a(n-1)$ and $a(n-1) = a(n-2) + 1$. At this case,

$$\begin{aligned} a(n+1) - a(n) &= a(n + 2 - 2 \cdot a(n)) - a(n - 2 \cdot a(n-2)) \\ &= a(n + 2 - 2 \cdot (a(n-2) + 1)) - a(n - 2 \cdot a(n-2)) \\ &= a(n - 2 \cdot a(n-2)) - a(n - 2 \cdot a(n-2)) \\ &= 0. \end{aligned}$$

Case 3. $a(n) = a(n-1)$ and $a(n-1) = a(n-2)$. At this case,

$$\begin{aligned} a(n+1) - a(n) &= a(n + 2 - 2 \cdot a(n)) - a(n - 2 \cdot a(n-2)) \\ &= a(n + 2 - 2 \cdot a(n)) - a(n - 2 \cdot a(n)) \end{aligned}$$

In here, we can use the fact that existence of two consecutive 1 in first differences is impossible because there is 2 difference in indices and this means that $a(n+2-2 \cdot a(n)) - a(n - 2 \cdot a(n)) \in \{0, 1\}$ by initial assumption. This case completes the induction. $\square$