Solution to Problem 11901 in *The American Mathematical Monthly*

Jerrold W. Grossman and Serge Kruk
Department of Mathematics and Statistics
Oakland University
Rochester, MI 48309
grossman@oakland.edu, 248-370-3443
August 21, 2016

A mapping f from $[n]$ to $[n]$ can be described by the vector $[f(1), f(2), \ldots, f[n]$. We claim that such a vector is a (possibly empty) composition of $\uparrow$ and $\downarrow$ if and only if it is constant or else of the form

$$[a, a, \ldots, a, a+1, a+2, \ldots, b, b, \ldots, b] \qquad (*)$$

with r appearances of a at the beginning and s appearances of b at the end, where $1 \leq a < b \leq n$ and $r + (b - a - 1) + s = n$. The constant vector $[c, c, c, \ldots, c]$ can be reached from the identity function $[1, 2, 3, \ldots, n]$ by $n-1$ applications of $\downarrow$ followed by $c-1$ applications of $\uparrow$. The vector $(*)$ can be reached from $[1, 2, 3, \ldots, n]$ by $r-1$ applications of $\downarrow$ to get a vector starting with r 1s, followed by $r+s-2$ applications of $\uparrow$ to get a vector ending with s ns, followed by $n-b$ applications of $\downarrow$. Conversely, it is clear that applying $\uparrow$ or $\downarrow$ to a vector of given form (or constant) leaves it in this form (or constant).

It remains to count the number of vectors of form $(*)$. For a given a and b, there are $n-b+a$ choices for r, and each such choice determines s. Taking into account the constant vectors as well gives the answer:

$$n + \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} (n - b + a) = \frac{2n^3 - 3n^2 + 7}{6}.$$

These values $(1, 3, 8, 18, 35, 61, 98, 148, 213, \ldots)$ form sequence A081489 in Neil Sloane's *On-line Encyclopedia of Integer Seqences* (oeis.org).