Derivation of a Recursive Formula

Let $S_n = \sum_{k=0}^{\infty}\left(\frac{k^n}{2^k}\right)$, where n ≥ 0. The sum of this series can be found by using the binomial expansion:

$$S_n + \binom{n}{n-1}\sum_{k=0}^{\infty}\left(\frac{k^{n-1}}{2^k}\right) + \binom{n}{n-2}\sum_{k=0}^{\infty}\left(\frac{k^{n-2}}{2^k}\right) + \cdots + \binom{n}{1}\sum_{k=0}^{\infty}\left(\frac{k^1}{2^k}\right) + \binom{n}{0}\sum_{k=0}^{\infty}\left(\frac{k^0}{2^k}\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{k^n}{2^k}\right) + \binom{n}{n-1}\sum_{k=0}^{\infty}\left(\frac{k^{n-1}}{2^k}\right) + \binom{n}{n-2}\sum_{k=0}^{\infty}\left(\frac{k^{n-2}}{2^k}\right) + \cdots + \binom{n}{1}\sum_{k=0}^{\infty}\left(\frac{k^1}{2^k}\right) + \binom{n}{0}\sum_{k=0}^{\infty}\left(\frac{k^0}{2^k}\right)$$

$$S_n + \binom{n}{n-1}S_{n-1} + \binom{n}{n-2}S_{n-2} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= \sum_{k=0}^{\infty}\left(\frac{k^n}{2^k}\right) + \sum_{k=0}^{\infty}\left(\binom{n}{n-1}\frac{k^{n-1}}{2^k}\right) + \sum_{k=0}^{\infty}\left(\binom{n}{n-2}\frac{k^{n-2}}{2^k}\right) + \cdots + \sum_{k=0}^{\infty}\left(\binom{n}{1}\frac{k^1}{2^k}\right)$$
$$+ \sum_{k=0}^{\infty}\left(\binom{n}{0}\frac{k^0}{2^k}\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{k^n}{2^k} + \binom{n}{n-1}\frac{k^{n-1}}{2^k} + \binom{n}{n-2}\frac{k^{n-2}}{2^k} + \cdots + \binom{n}{1}\frac{k^1}{2^k} + \binom{n}{0}\frac{k^0}{2^k}\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{2^k}\left(k^n + \binom{n}{n-1}k^{n-1} + \binom{n}{n-2}k^{n-2} + \cdots + \binom{n}{1}k^1 + \binom{n}{0}k^0\right)\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{2^k}(k+1)^n\right) = \sum_{k=0}^{\infty}\left(\frac{(k+1)^n}{2^k}\right) = 2\sum_{k=0}^{\infty}\left(\frac{(k+1)^n}{2(2^k)}\right) = 2\sum_{k=0}^{\infty}\left(\frac{(k+1)^n}{2^{k+1}}\right)$$

If we substitute k+1 with the variable i, we will discover that the new series is the same as the original series, but with the first term missing:

$$\sum_{k=0}^{\infty}\left(\frac{(k+1)^n}{2^{k+1}}\right) = \sum_{i=1}^{\infty}\left(\frac{i^n}{2^i}\right)$$

However, the first term of the series equals 0, so we can add it back without changing the value:

$$\sum_{i=1}^{\infty}\left(\frac{i^n}{2^i}\right) = \sum_{i=1}^{\infty}\left(\frac{i^n}{2^i}\right) + \frac{0^n}{2^0} - \frac{0^n}{2^0} = \sum_{i=0}^{\infty}\left(\frac{i^n}{2^i}\right) - \frac{0}{1} = \sum_{i=0}^{\infty}\left(\frac{i^n}{2^i}\right) - 0 = \sum_{i=0}^{\infty}\left(\frac{i^n}{2^i}\right) = S_n$$

If we substitute this back into the original equation, we have a formula for S_n in terms of S_{n-1}, S_{n-2}, …, S_1, S_0:

$$S_n + \binom{n}{n-1}S_{n-1} + \binom{n}{n-2}S_{n-2} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0 = 2S_n$$

$$S_n = \binom{n}{n-1}S_{n-1} + \binom{n}{n-2}S_{n-2} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

The value of S_n can be determined by calculating the value of S_{n-1}, S_{n-2}, …, S_1, S_0 and substituting them into the recursive formula. However, in order to calculate S_{n-1}, we need to use the recursive formula and substitute the values of S_{n-2}, S_{n-3}, …, S_1, S_0. In other words, we need to run the formula once for S_n and n-1 times for all the intermediate values. The values of S_n calculated for several values of n (starting at n=0) are:

2, 2, 6, 26, 150, 1082, 9366, 94586, 1091670, 14174522, 204495126, 3245265146, 56183135190, 1053716696762, 21282685940886, 460566381955706, 10631309363962710, 260741534058271802, 6771069326513690646, 185603174638656822266

The above values came from the Online Encyclopedia of Integer Sequences, entry A076726. If we can find a non-recursive formula for S_n, then we can find its value more efficiently.

Derivation of a Non-Recursive Formula

The first step towards deriving a non-recursive formula for S_n involves replacing S_{n-1} with its recursive definition:

$$S_n = \binom{n}{n-1}S_{n-1} + \binom{n}{n-2}S_{n-2} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= \binom{n}{n-1}\left(\binom{n-1}{n-2}S_{n-2} + \binom{n-1}{n-3}S_{n-3} + \cdots + \binom{n-1}{1}S_1 + \binom{n-1}{0}S_0\right)$$

$$+ \binom{n}{n-2}S_{n-2} + \binom{n}{n-3}S_{n-3} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= \binom{n}{n-1}\binom{n-1}{n-2}S_{n-2} + \binom{n}{n-1}\binom{n-1}{n-3}S_{n-3} + \cdots + \binom{n}{n-1}\binom{n-1}{1}S_1$$

$$+ \binom{n}{n-1}\binom{n-1}{0}S_0$$

$$+ \binom{n}{n-2}S_{n-2} + \binom{n}{n-3}S_{n-3} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= \frac{n!}{(n-1)!\,1!}\frac{(n-1)!}{(n-2)!\,1!}S_{n-2} + \frac{n!}{(n-1)!\,1!}\frac{(n-1)!}{(n-3)!\,2!}S_{n-3} + \cdots + \frac{n!}{(n-1)!\,1!}\frac{(n-1)!}{1!\,(n-2)!}S_1$$

$$+ \frac{n!}{(n-1)!\,1!}\frac{(n-1)!}{0!\,(n-1)!}S_0$$

$$+ \binom{n}{n-2}S_{n-2} + \binom{n}{n-3}S_{n-3} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= \frac{1}{1!}\frac{n!}{(n-2)!\,1!}S_{n-2} + \frac{1}{1!}\frac{n!}{(n-3)!\,2!}S_{n-3} + \cdots + \frac{1}{1!}\frac{n!}{1!\,(n-2)!}S_1 + \frac{1}{1!}\frac{n!}{0!\,(n-1)!}S_0$$

$$+ \binom{n}{n-2}S_{n-2} + \binom{n}{n-3}S_{n-3} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= \frac{2}{1}\frac{n!}{(n-2)!\,2(1!)}S_{n-2} + \frac{3}{1}\frac{n!}{(n-3)!\,3(2!)}S_{n-3} + \cdots + \frac{n-1}{1}\frac{n!}{1!\,(n-1)(n-2)!}S_1$$

$$+ \frac{n}{1}\frac{n!}{0!\,n(n-1)!}S_0$$

$$+ \binom{n}{n-2}S_{n-2} + \binom{n}{n-3}S_{n-3} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= 2\frac{n!}{(n-2)!\,2!}S_{n-2} + 3\frac{n!}{(n-3)!\,3!}S_{n-3} + \cdots + n-1\frac{n!}{1!\,(n-1)!}S_1 + n\frac{n!}{0!\,n!}S_0$$

$$+ \binom{n}{n-2}S_{n-2} + \binom{n}{n-3}S_{n-3} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= 2\binom{n}{n-2}S_{n-2} + 3\binom{n}{n-3}S_{n-3} + \cdots + (n-1)\binom{n}{1}S_1 + n\binom{n}{0}S_0$$

$$+ \binom{n}{n-2}S_{n-2} + \binom{n}{n-3}S_{n-3} + \cdots + \binom{n}{1}S_1 + \binom{n}{0}S_0$$

$$= (2+1)\binom{n}{n-2}S_{n-2} + (3+1)\binom{n}{n-3}S_{n-3} + \cdots + ((n-1)+1)\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= 3\binom{n}{n-2}S_{n-2} + 4\binom{n}{n-3}S_{n-3} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

This formula provides a definition for S_n in terms of S_{n-2}, S_{n-3}, …, S_1, S_0. It is still recursive, which means we have to run it multiple times to get the intermediate terms. However, it provides an interesting result if we substitute 2 for n:

$$S_2 = 3\binom{2}{2-2}S_{2-2} + 4\binom{2}{2-3}S_{2-3} + \cdots + 2\binom{2}{1}S_1 + (2+1)\binom{2}{0}S_0$$

$$= 3\binom{2}{0}S_0 + 4\binom{2}{-1}S_{-1} + \cdots + 3(1)S_0$$

All the terms except for the first one disappear, since they go back past S_0. Therefore, the only term that remains is the first one:

$$S_2 = 3(1)S_0 = 3S_0$$

In other words, the value of S_2 is three times that of S_0, giving us a result of 6.

The second step towards deriving a non-recursive formula involves replacing S_{n-2} with its recursive definition:

$$S_n = 3\binom{n}{n-2}\left(\binom{n-2}{n-3}S_{n-3} + \binom{n-2}{n-4}S_{n-4} + \cdots + \binom{n-2}{1}S_1 + \binom{n-2}{0}S_0\right)$$

$$+4\binom{n}{n-3}S_{n-3} + 5\binom{n}{n-4}S_{n-4} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= 3\binom{n}{n-2}\binom{n-2}{n-3}S_{n-3} + 3\binom{n}{n-2}\binom{n-2}{n-4}S_{n-4} + \cdots + 3\binom{n}{n-2}\binom{n-2}{1}S_1$$

$$+ 3\binom{n}{n-2}\binom{n-2}{0}S_0$$

$$+4\binom{n}{n-3}S_{n-3} + 5\binom{n}{n-4}S_{n-4} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= 3\frac{n!}{(n-2)!\,2!}\frac{(n-2)!}{(n-3)!\,1!}S_{n-3} + 3\frac{n!}{(n-2)!\,2!}\frac{(n-2)!}{(n-4)!\,2!}S_{n-4} + \cdots$$

$$+ 3\frac{n!}{(n-2)!\,2!}\frac{(n-2)!}{1!\,(n-3)!}S_1 + 3\frac{n!}{(n-2)!\,2!}\frac{(n-2)!}{0!\,(n-2)!}S_0$$

$$+4\binom{n}{n-3}S_{n-3} + 5\binom{n}{n-4}S_{n-4} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= \frac{3}{2!}\frac{n!}{(n-3)!\,1!}S_{n-3} + \frac{3}{2!}\frac{n!}{(n-4)!\,2!}S_{n-4} + \cdots + \frac{3}{2!}\frac{n!}{1!\,(n-3)!}S_1 + \frac{3}{2!}\frac{n!}{0!\,(n-2)!}S_0$$

$$+4\binom{n}{n-3}S_{n-3} + 5\binom{n}{n-4}S_{n-4} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= \frac{3*3*2}{2!}\frac{n!}{(n-3)!\,3(2)(1!)}S_{n-3} + \frac{3*4*3}{2!}\frac{n!}{(n-4)!\,4(3)(2!)}S_{n-4} + \cdots$$

$$+ \frac{3*(n-1)*(n-2)}{2!}\frac{n!}{1!\,(n-1)(n-2)(n-3)!}S_1$$

$$+ \frac{3*n*(n-1)}{2!}\frac{n!}{0!\,n(n-1)(n-2)!}S_0$$

$$+4\binom{n}{n-3}S_{n-3} + 5\binom{n}{n-4}S_{n-4} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= 9\frac{n!}{(n-3)!\,3!}S_{n-3} + 18\frac{n!}{(n-4)!\,4!}S_{n-4} + \cdots + \frac{3*(n^2-n-2n+2)}{2}\frac{n!}{1!\,(n-1)!}S_1 + \frac{3*(n^2-n)}{2}\frac{n!}{0!\,n!}S_0$$

$$+4\binom{n}{n-3}S_{n-3} + 5\binom{n}{n-4}S_{n-4} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= 9\binom{n}{n-3}S_{n-3} + 18\binom{n}{n-4}S_{n-4} + \cdots + \frac{3*(n^2-3n+2)}{2}\binom{n}{1}S_1 + \frac{3*(n^2-n)}{2}\binom{n}{0}S_0$$

$$+4\binom{n}{n-3}S_{n-3} + 5\binom{n}{n-4}S_{n-4} + \cdots + n\binom{n}{1}S_1 + (n+1)\binom{n}{0}S_0$$

$$= 13\binom{n}{n-3}S_{n-3} + 23\binom{n}{n-4}S_{n-4} + \cdots + \left(\frac{3n^2-9n+6}{2} + n\right)\binom{n}{1}S_1 + \left(\frac{3n^2-3n}{2} + n + 1\right)\binom{n}{0}S_0$$

$$= 13\binom{n}{n-3}S_{n-3} + 23\binom{n}{n-4}S_{n-4} + \cdots + \left(\frac{3n^2-9n+6}{2} + \frac{2n}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-3n}{2} + \frac{2(n+1)}{2}\right)\binom{n}{0}S_0$$

$$= 13\binom{n}{n-3}S_{n-3} + 23\binom{n}{n-4}S_{n-4} + \cdots + \left(\frac{3n^2-9n+6+2n}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-3n}{2} + \frac{2n+2}{2}\right)\binom{n}{0}S_0$$

$$= 13\binom{n}{n-3}S_{n-3} + 23\binom{n}{n-4}S_{n-4} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-3n+2n+2}{2}\right)\binom{n}{0}S_0$$

$$= 13\binom{n}{n-3}S_{n-3} + 23\binom{n}{n-4}S_{n-4} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-n+2}{2}\right)\binom{n}{0}S_0$$

This formula provides a definition for S_n in terms of S_{n-3}, S_{n-4}, …, S_1, S_0. It is still recursive, which means we have to run it multiple times to get the intermediate terms. However, it provides an interesting result if we substitute 3 for n:

$$S_3 = 13\binom{3}{3-3}S_{3-3} + 23\binom{3}{3-4}S_{3-4} + \cdots + \left(\frac{3(3^2)-3+2}{2}\right)\binom{3}{0}S_0$$

$$= 13\binom{3}{0}S_0 + 23\binom{3}{-1}S_{-1} + \cdots + \left(\frac{3(9)-1}{2}\right)(1)S_0$$

All the terms after the first one disappear, since they go back past S_0. Therefore, the only term that remains is the S_0 term:

$S_3 = 13(1)S_0 = 13S_0$

In other words, the value of S_3 can be found by multiplying the value of S_0 by a 13, giving us a result of 26.

The third step towards deriving a non-recursive formula involves replacing S_{n-3} with its recursive definition:

$$S_n = 13\binom{n}{n-3}S_{n-3} + 23\binom{n}{n-4}S_{n-4} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-n+2}{2}\right)\binom{n}{0}S_0$$

$$= 13\binom{n}{n-3}\left(\binom{n-3}{n-4}S_{n-4} + \binom{n-3}{n-5}S_{n-5} + \cdots + \binom{n-3}{1}S_1 + \binom{n-3}{0}S_0\right)$$

$$+23\binom{n}{n-4}S_{n-4} + \left(\frac{3(5^2)-5+2}{2}\right)\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-n+2}{2}\right)\binom{n}{0}S_0$$

$$= 13\binom{n}{n-3}\binom{n-3}{n-4}S_{n-4} + 13\binom{n}{n-3}\binom{n-3}{n-5}S_{n-5} + \cdots + 13\binom{n}{n-3}\binom{n-3}{1}S_1 + 13\binom{n}{n-3}\binom{n-3}{0}S_0$$

$$+23\binom{n}{n-4}S_{n-4} + \left(\frac{3(25)-3}{2}\right)\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-n+2}{2}\right)\binom{n}{0}S_0$$

$$= 13\frac{n!}{(n-3)!\,3!}\frac{(n-3)!}{(n-4)!\,1!}S_{n-4} + 13\frac{n!}{(n-3)!\,3!}\frac{(n-3)!}{(n-5)2!}S_{n-5} + \cdots + 13\frac{n!}{(n-3)!\,3!}\frac{(n-3)!}{1!\,(n-4)!}S_1 + 13\frac{n!}{(n-3)!\,3!}\frac{(n-3)!}{0!\,(n-3)!}S_0$$

$$+23\binom{n}{n-4}S_{n-4} + \left(\frac{75-3}{2}\right)\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-n+2}{2}\right)\binom{n}{0}S_0$$

$$= \frac{13}{3!}\frac{n!}{(n-4)!\,1!}S_{n-4} + \frac{13}{3!}\frac{n!}{(n-5)!\,2!}S_{n-5} + \cdots + \frac{13}{3!}\frac{n!}{1!\,(n-4)!}S_1 + \frac{13}{3!}\frac{n!}{0!\,(n-3)!}S_0$$

$$+23\binom{n}{n-4}S_{n-4} + \left(\frac{72}{2}\right)\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-n+2}{2}\right)\binom{n}{0}S_0$$

$$= \frac{13*4*3*2}{3!}\frac{n!}{(n-4)!\,4(3)(2)(1!)}S_{n-4} + \frac{13*5*4*3}{3!}\frac{n!}{(n-5)!\,5(4)(3)(2!)}S_{n-5} + \cdots + \frac{13*(n-1)*(n-2)*(n-3)}{3!}\frac{n!}{1!\,(n-1)(n-2)(n-3)(n-4)!}S_1 + \frac{13*n*(n-1)*(n-2)}{3!}\frac{n!}{0!\,n(n-1)(n-2)(n-3)!}S_0$$

$$+23\binom{n}{n-4}S_{n-4} + 36\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{3n^2-7n+6}{2}\right)\binom{n}{1}S_1 + \left(\frac{3n^2-n+2}{2}\right)\binom{n}{0}S_0$$

$$= 52\frac{n!}{(n-4)!\,4!}S_{n-4} + 130\frac{n!}{(n-5)!\,5!}S_{n-5} + \cdots$$
$$+ \frac{13*(n^2-2n-n+2)*(n-3)}{6}\frac{n!}{1!\,(n-1)!}S_1$$
$$+ \frac{13*(n^2-n)*(n-2)}{6}\frac{n!}{0!\,n!}S_0$$
$$+23\binom{n}{n-4}S_{n-4} + 36\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{3(3n^2-7n+6)}{3(2)}\right)\binom{n}{1}S_1$$
$$+ \left(\frac{3(3n^2-n+2)}{3(2)}\right)\binom{n}{0}S_0$$
$$= 52\binom{n}{n-4}S_{n-4} + 130\binom{n}{n-5}S_{n-5} + \cdots + \frac{13*(n^2-3n+2)*(n-3)}{6}\binom{n}{1}S_1$$
$$+ \frac{13*(n^3-n^2-2n^2+\ 2n)}{6}\binom{n}{0}S_0$$
$$+23\binom{n}{n-4}S_{n-4} + 36\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{9n^2-21n+18}{6}\right)\binom{n}{1}S_1$$
$$+ \left(\frac{9n^2-3n+6}{6}\right)\binom{n}{0}S_0$$
$$= 75\binom{n}{n-4}S_{n-4} + 166\binom{n}{n-5}S_{n-5} + \cdots + \frac{13*(n^3-3n^2+2n-3n^2+9n-6)}{6}\binom{n}{1}S_1$$
$$+ \frac{13*(n^3-3n^2+\ 2n)}{6}\binom{n}{0}S_0$$
$$+ \left(\frac{9n^2-21n+18}{6}\right)\binom{n}{1}S_1 + \left(\frac{9n^2-3n+6}{6}\right)\binom{n}{0}S_0$$
$$= 75\binom{n}{n-4}S_{n-4} + 166\binom{n}{n-5}S_{n-5} + \cdots$$
$$+ \left(\frac{13*(n^3-6n^2+11n-6)}{6} + \frac{9n^2-21n+18}{6}\right)\binom{n}{1}S_1$$
$$+ \left(\frac{13n^3-39n^2+\ 26n}{6} + \frac{9n^2-3n+6}{6}\right)\binom{n}{0}S_0$$
$$= 75\binom{n}{n-4}S_{n-4} + 166\binom{n}{n-5}S_{n-5} + \cdots$$
$$+ \left(\frac{13n^3-78n^2+143n-78}{6} + \frac{9n^2-21n+18}{6}\right)\binom{n}{1}S_1$$
$$+ \left(\frac{13n^3-39n^2+\ 26n+9n^2-3n+6}{6}\right)\binom{n}{0}S_0$$
$$= 75\binom{n}{n-4}S_{n-4} + 166\binom{n}{n-5}S_{n-5} + \cdots$$
$$+ \left(\frac{13n^3-78n^2+143n-78+9n^2-21n+18}{6}\right)\binom{n}{1}S_1$$
$$+ \left(\frac{13n^3-30n^2+\ 23n+6}{6}\right)\binom{n}{0}S_0$$

$$= 75\binom{n}{n-4}S_{n-4} + 166\binom{n}{n-5}S_{n-5} + \cdots + \left(\frac{13n^3 - 69n^2 + 122n - 60}{6}\right)\binom{n}{1}S_1 + \left(\frac{13n^3 - 30n^2 + 23n + 6}{6}\right)\binom{n}{0}S_0$$

This formula provides a definition for S_n in terms of S_{n-4}, S_{n-5}, …, S_1, S_0. It is still recursive, which means we have to run it multiple times to get the intermediate terms. However, it provides an interesting result if we substitute 4 for n:

$$S_4 = 75\binom{4}{4-4}S_{4-4} + 166\binom{4}{4-5}S_{4-5} + \cdots + \left(\frac{13(4^3) - 30(4^2) + 23(4) + 6}{6}\right)\binom{4}{0}S_0$$

$$S_4 = 75\binom{4}{0}S_0 + 166\binom{4}{-1}S_{-1} + \cdots + \left(\frac{13(64) - 30(16) + 92 + 6}{6}\right)(1)S_0$$

All the terms after the first one disappear, since they go back past S_0. Therefore, the only term that remains is the S_0 term:

$$S_3 = 75(1)S_0 = 75S_0$$

In other words, the value of S_4 can be found by multiplying the value of S_0 by a 75, giving us a result of 150. Each time we perform a substitution, the first coefficient of the recursive formula provides us with a value that we can multiply S_0 with to get a further term in the sequence. Now, I will try to perform a more general substitution, and see if it gives me a formula for S_n solely in terms of S_0.

Derivation of the Polynomial Formula

The process of generating values for S_1, S_2, S_3, …, S_k has a regular pattern. First, we express S_n using a recursive formula that contains the terms S_{n-k}, S_{n-k-1}, …, S_1, S_0. The intermediate terms will have a coefficient that equals some constant value multiplied by a combination:

$$S_n = C_{n-k}\binom{n}{n-k}S_{n-k} + C_{n-k-1}\binom{n}{n-k-1}S_{n-k-1} + \cdots + C_1\binom{n}{1}S_1 + C_0\binom{n}{0}S_0$$

The coefficient for C_0 is usually a polynomial, which we will refer to as the "generating polynomial" for k. The coefficient of the intermediate term S_j can be found by substituting n-j into the generating polynomial:

$$S_n = GP_k(n-k)\binom{n}{n-k}S_{n-k} + GP_k(n-k-1)\binom{n}{n-k-1}S_{n-k-1} + \cdots + GP_k(n-1)\binom{n}{1}S_1 + GP_k(n)\binom{n}{0}S_0$$

Substituting n-k into the generating polynomial gives us a value that we can multiply by S_0 to get S_k. In order to find the generating polynomial for k+1, we must substitute S_{n-k} with its recursive definition:

$$S_n = GP_k(n-k)\binom{n}{n-k}\left(\binom{n-k}{n-k-1}S_{n-k-1} + \binom{n-k}{n-k-2}S_{n-k-2} + \cdots + \binom{n-k}{1}S_1 + \binom{n-k}{0}S_0\right) + GP_k(n-k-1)\binom{n}{n-k-1}S_{n-k-1} + \cdots + GP_k(n-1)\binom{n}{1}S_1 + GP_k(n)\binom{n}{0}S_0$$

If we expand this equation, we will get a recursive formula for S_n that contains the intermediate terms S_{n-k-1}, S_{n-k-2}, …, S_1, S_0:

$$S_n = GP_k(n-k)\binom{n}{n-k}\binom{n-k}{n-k-1}S_{n-k-1} + GP_k(n-k)\binom{n}{n-k}\binom{n-k}{n-k-2}S_{n-k-2} + \cdots$$
$$+ GP_k(n-k)\binom{n}{n-k}\binom{n-k}{1}S_1 + GP_k(n-k)\binom{n}{n-k}\binom{n-k}{0}S_0$$
$$+GP_k(n-k-1)\binom{n}{n-k-1}S_{n-k-1} + \cdots + GP_k(n-1)\binom{n}{1}S_1 + GP_k(n)\binom{n}{0}S_0$$

The coefficient of the S_{n-k-1} term gives us the generating polynomial for S_{k+1}, because when we substitute k+1 for n, the remaining terms end up going back past S_0, leaving only the S_{n-k-1} term.

$$GP_{k+1}(n) = GP_k(n-k)\frac{n!}{(n-k)!\,k!}\frac{(n-k)!}{(n-k-1)!\,1!} + GP_k(n-k-1)\binom{n}{n-k-1}$$
$$= \frac{GP_k(n-k)}{k!}\frac{n!}{(n-k-1)!\,1!} + GP_k(n-k-1)\binom{n}{n-k-1}$$
$$= \frac{GP_k(n-k)*(k+1)*k*(k-1)*\ldots*3*2}{k!} * \frac{n!}{(k+1)k(k-1)*\ldots*3(2)1!\,(n-k-1)!}$$
$$+ GP_k(n-k-1)\binom{n}{n-k-1}$$
$$= \frac{GP_k(n-k)*(k+1)!}{k!} * \frac{n!}{(k+1)!\,(n-k-1)!} + GP_k(n-k-1)\binom{n}{n-k-1}$$
$$= GP_k(n-k)*(k+1)*\binom{n}{n-k-1} + GP_k(n-k-1)\binom{n}{n-k-1}$$
$$= (GP_k(n-k)*(k+1) + GP_k(n-k-1))*\binom{n}{n-k-1}$$

This recursive formula allows us to recursively determine the generating polynomial for S_{k+1} using the generating polynomial for S_k. For our base case, we can assume that $GP_0 = 1$, since $S_0 = 1*S_0$.

The value of S_{k+1} can be found by substituting k+1 into the generating polynomial:

$$S_{k+1}/S_0 = (GP_k(k+1-k)*(k+1) + GP_k(k+1-k-1))*\binom{k+1}{k+1-k-1}$$
$$= (GP_k(1)*(k+1) + GP_k(0))*\binom{k+1}{0}$$
$$= (GP_k(1)*(k+1) + GP_k(0))*1$$
$$= (k+1)GP_k(1) + GP_k(0)$$

The value obtained by evaluating the generating polynomial of k at n=0 and n=1 must be multiplied by S_0 in order to obtain S_{k+1}.