Gamma of reciprocal

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Abstract

This document gives a definition of $\Gamma(\frac{1}{m})$ using Laplace transform.

1 General formula

$$\frac{\Gamma(x)}{s^x} = \int_0^\infty t^{x-1}e^{-st}dt\,; \quad s = 1 \Rightarrow$$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \Rightarrow x\Gamma(x) = \Gamma(x+1) = \int_0^\infty t^x e^{-t}dt$$

$$t = u^m \Rightarrow \frac{dt}{du} = mu^{m-1} \Rightarrow dt = mu^{m-1}du$$

$$\Gamma(x+1) = \int_0^\infty (u^m)^x e^{-u^m} mu^{m-1}du = m\int_0^\infty \frac{1}{u}u^m u^{xm} e^{-u^m}du = m\int_0^\infty \frac{1}{u}u^{m(x+1)}e^{-u^m}du$$

$$x+1 = \frac{n+1}{m} \Rightarrow \Gamma(x+1) = \Gamma\Big(\frac{n+1}{m}\Big) = m\int_0^\infty \frac{1}{u}u^{\frac{m(n+1)}{m}}e^{-u^m}du = m\int_0^\infty u^n e^{-u^m}du$$

Summing up:

$$\Gamma\Big(\frac{n+1}{m}\Big) = m\int_0^\infty u^n e^{-u^m}du\,; \quad \Gamma\Big(\frac{1}{m}\Big) = m\int_0^\infty e^{-u^m}du$$

Dwight 860.19 , 860.18

2 $\Gamma(\frac{1}{3})$

$$m = 3 \Rightarrow t = u^3 \Rightarrow \frac{dt}{du} = 3u^2 \Rightarrow dt = 3u^2du$$

$$x\Gamma(x) = \Gamma(x+1) = \int_0^\infty (u^3)^x e^{-u^3}3u^2du = 3\int_0^\infty \frac{1}{u}u^3u^{3x}e^{-u^3}du =$$

$$\Gamma(x+1) = 3\int_0^\infty \frac{1}{u}u^{3x+3}e^{-u^3}du = 3\int_0^\infty \frac{1}{u}u^{3(x+1)}e^{-u^3}du\,; \quad x+1 = \frac{n+1}{3} \Rightarrow$$

$$\Gamma\Big(\frac{n+1}{3}\Big) = 3\int_0^\infty \frac{1}{u}u^{3(\frac{n+1}{3})}e^{-u^3}du = 3\int_0^\infty \frac{1}{u}u^{n+1}e^{-u^3}du = 3\int_0^\infty u^n e^{-u^3}du$$

$$\Gamma\Big(\frac{n+1}{3}\Big) = 3\int_0^\infty u^n e^{-u^3}du\,; \quad n = 0 \Rightarrow \qquad \Gamma\Big(\frac{1}{3}\Big) = 3\int_0^\infty e^{-u^3}du$$

2.1 Verification

$$s_1 = \frac{3}{10^3}\sum_{k=0}^{1000} e^{-j^3} = 2.422533684373850...;\ s_2 = \frac{3}{10^3}\sum_{k=1000}^{2000} e^{-j^3} = 0.256326859010989...$$

$$s_3 = \frac{3}{10^3}\sum_{k=2000}^{3000} e^{-j^3} = 0.000077991322722...;\ s_4 = \frac{3}{10^3}\sum_{k=3000}^{4000} e^{-j^3} = 0.000000000000204...$$

where j=k/1000+0.0005
$\Gamma(\frac{1}{3}) \approx s_1 + s_2 + s_3 + s_4 = 2.678938534707765...$

3 Source of data

With $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (A002161) and $\Gamma(\frac{1}{3})$ = A073005 from oeis.org we get these values:
$\Gamma(\frac{1}{4}) = A068466\,;\ \Gamma(\frac{1}{5}) = A175380\,;\ \Gamma(\frac{1}{6}) \ = A175379\,;\ \Gamma(\frac{1}{7}) \ = A220086\,;$
$\Gamma(\frac{1}{8}) = A203142\,;\ \Gamma(\frac{1}{9}) = A256190\,;\ \Gamma(\frac{1}{10}) = A256191\,;\ \Gamma(\frac{1}{11}) = A256192\,;...$

3.1 Values

m	$\Gamma(1/m)$	m	$\Gamma(1/m)$	m	$\Gamma(1/m)$
1	1	18	17.475090507153	35	34.4503245400162
2	1.77245385090552	19	18.4724620970425	36	35.4495783705751
3	2.67893853470775	20	19.4700853112555	37	36.4488715722962
4	3.62560990822191	21	20.4679256534386	38	37.4482011088367
5	4.59084371199881	22	21.4659546597143	39	38.4475642482838
6	5.56631600178024	23	22.4641486347938	40	39.4469585259301
7	6.54806294024783	24	23.4624876931833	41	40.4463817123814
8	7.53394159879761	25	24.4609550228561	42	41.4458317861256
9	8.52268813921948	26	25.4595363133927	43	42.4453069098525
10	9.51350769866873	27	26.4582193072568	44	43.4448054099386
11	10.5058748560787	28	27.4569934443372	45	44.4443257586139
12	11.499428186074	29	28.455849577892	46	45.4438665584107
13	12.4939108170965	30	29.4547797456997	47	46.4434265285612
14	13.4891351302741	31	30.4537769842894	48	47.4430044930635
15	14.4849608719646	32	31.4528351770761	49	48.442599370184
16	15.4812810815924	33	32.4519489293925	50	49.4422101631957
17	16.4780127590935	34	33.4511134650191	51	50.4418359521893

3.2 Bibliography

Herbert B. Dwight - *Tables of Integrals and other Mathematical Data* - Macmillan, New York - 1961
Eric Weisstein - *https://mathworld.wolfram.com/GammaFunction.html*
On-Line Encyclopedia of Integer Sequences - https://oeis.org