On the external path length of a Fibonacci tree

John M. Campbell
jmaxwellcampbell@gmail.com

Abstract

We prove a new immanantal formula for the external path length $\text{EPL}(T_n)$ of a Fibonacci tree T_n, and we use this formula to prove the new binomial identity $\text{EPL}(T_n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}(n-2i-1)$.

1 Introduction

The *external path length* $\text{EPL}(T)$ of a binary tree T is defined so that

$$\text{EPL}(T) = \sum_{\substack{\lambda \text{ is the level} \\ \text{of an external node}}} \lambda.$$

The Fibonacci tree T_n of order $n \in \mathbb{N}$ is defined inductively as follows: for $n \in \{1, 2\}$, T_n consists of a single root, and for $n \geq 3$, the left subtree of T_n is T_{n-1}, and the right subtree of T_n is T_{n-2}. In this paper, we prove some new formulas for the external path length of a Fibonacci tree.

Letting $n \geq 3$, it is easily seen that the external path length $\text{EPL}(T_n)$ of T_n is equal to the convolution $\sum_{k=0}^{n-3} F_{k+1}F_{n-k}$ given in the OEIS sequence A067331 [2]. Interestingly, this integer sequence is also related to tilings of "triangular strips" with triangles as indicated in [1]. Since the OEIS sequence A067331 is related to combinatorial objects such as Fibonacci trees and triangular tilings, it seems natural to consider other combinatorial properties associated with this sequence. In particular, we show how the external path length of a Fibonacci tree may be expressed in a natural way using a class of hook immanants.

The immanant of a matrix A of order n corresponding to the hook partition $(k, 1^{n-k}) \vdash n$ is denoted by d_k. We refer to the hook immanant d_{n-1} as the *penultimate immanant*. In Section 2, we prove that:

$$\text{EPL}(T_n) = d_{n-1} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \ddots & 0 \\ 0 & 1 & 1 & 1 & \ddots & 0 \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}_{n \times n}. \tag{1.1}$$

We discovered this elegant formula using the OEIS [2]. We prove (1.1) using elementary character theory, and our proof of (1.1) may be reformulated in a natural way using border-strip tableaux. This provides another combinatorial property associated with the external path length of a Fibonacci tree.

The equality given in (1.1) is interesting because (1.1) suggests an unexpected representation-theoretic interpretation of the external path length of a Fibonacci tree. In particular, from (1.1), we have that $\text{EPL}(T_n)$ may be expressed in a simple and unexpected way as a character sum as follows:

$$\text{EPL}(T_n) = \sum_{\substack{\sigma \in S_n \\ \forall i \ |\sigma_i - i| \leq 1}} \chi^{(n-1,1)}(\sigma). \tag{1.2}$$

We use (1.2) to construct an unexpected and elegant representation-theoretic proof of the following new binomial identity:

$$\mathrm{EPL}(T_n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}(n-2i-1). \tag{1.3}$$

In Section 1.1, we briefly review basic terminology and notation related to immanants. In Section 2, we prove a recursive formula for the penultimate immanant of an arbitrary tridiagonal matrix with constant diagonals, and we use this recursive formula to prove (1.1). In Section 3, we present a proof of the identity given in (1.3).

1.1 The penultimate immanant

Letting λ be a partition of $n \in \mathbb{N}$, and letting χ_λ denote the corresponding irreducible representation-theoretic character of S_n, the *immanant* of an $n \times n$ matrix $A = (a_{i,j})_{n \times n}$ corresponding to the partition λ is defined as follows:

$$\mathrm{Imm}^{\lambda}(A) = \sum_{\sigma \in S_n} \chi^{\lambda}(\sigma) a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}.$$

For $\sigma \in S_n$, let $\mathrm{cycle}(\sigma) \vdash n$ denote the cycle type of σ. We thus have that:

$$\mathrm{Imm}^{\lambda}(A) = \sum_{\sigma \in S_n} \chi^{\lambda}_{\mathrm{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}. \tag{1.4}$$

Let $\lambda \vdash n$ and $\rho \vdash n$. By the Murnaghan-Nakayama rule, we have that

$$\chi^{\lambda}_{\rho} = \sum_{T} (-1)^{\mathrm{ht}(T)}$$

where the sum is over all border-strip tableaux of shape λ and content ρ, and $\mathrm{ht}(T)$ denotes the sum of the heights of the border strips in T. The height of a border strip is -1 plus the number of rows it touches. A border-strip tableau of shape λ is a tableau with weakly increasing rows and columns such that: for all indices i, the arrangement of cells labeled i forms a contiguous border strip.

Recall that the *penultimate immanant* of an $n \times n$ matrix A is the immanant of A corresponding to the partition $(n-1, 1)$. We use the notation PenImm to denote the penultimate immanant. Irreducible characters of the form $\chi^{(n-1,1)}_{\rho}$ may be evaluated in a natural way using the Murnaghan-Nakayama rule.

Example 1.1. We adopt the convention whereby the first entry of a partition is illustrated at the bottom of the corresponding diagram. Letting $\rho = (3, 2, 2, 1, 1)$, the border-strip tableaux corresponding to the character $\chi^{(8,1)}_{\rho}$ are given below:

1							
1	1	2	2	3	3	4	5

5							
1	1	1	2	2	3	3	4

4							
1	1	1	2	2	3	3	5

We thus have that $\chi^{(8,1)}_{\rho} = (-1)^1 + (-1)^0 + (-1)^0 = 1$.

Using the Murnaghan-Nakayama rule, it is easily verified that:

Claim 1.2. For $\sigma \in S_n$, $\chi^{(n-1,1)}_{\mathrm{cycle}(\sigma)} = (\#\text{ of fixed points of } \sigma) - 1$.

2 The external path length of a Fibonacci tree

The $n \times n$ tridiagonal matrix with α's along the subdiagonal, β's along the main diagonal, and γ's along the superdiagonal is henceforward denoted as $\text{trid}_n^{\alpha,\beta,\gamma}$, or simply trid_n for the sake of convenience.

Lemma 2.1. *The sequence* $\left(d_{n-1}\left(\text{trid}_n^{\alpha,\beta,\gamma}\right) : n \geq 3\right)$ *is generated by the linear recurrence given by* $(2\beta, 2\alpha\gamma - \beta^2, -2\alpha\beta\gamma, -(\alpha\gamma)^2)$.

Proof. Write $\text{trid}_n^{\alpha,\beta,\gamma} = \text{trid}_n = (a_{i,j})_{n\times n}$. From (1.4), we have that:

$$d_{n-1}(\text{trid}_n) = \sum_{\sigma \in S_n} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}. \tag{2.1}$$

Consider the expression $\text{perm}(\text{trid}_n)$. It is clear that if the product

$$a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}$$

given in (2.1) does not vanish, then σ is such that $|\sigma_i - i| \leq 1$ for all indices i. Let R_n denote the collection of all permutations in S_n of this form. From (2.1), we thus have that:

$$d_{n-1}(\text{trid}_n) = \sum_{\sigma \in R_n} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n-1,\sigma_{n-1}} a_{n,\sigma_n}. \tag{2.2}$$

If the expression a_{n,σ_n} does not vanish, then $\sigma_n \in \{n-1, n\}$. From (2.2), we thus have that $d_{n-1}(\text{trid}_n)$ is equal to:

$$\sum_{\substack{\sigma \in R_n \\ \sigma_n = n-1}} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n-1,\sigma_{n-1}} a_{n,n-1} +$$
$$\sum_{\substack{\sigma \in R_n \\ \sigma_n = n}} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n-1,\sigma_{n-1}} a_{n,n}.$$

Now observe that given a permutation σ such that $|\sigma_i - i| \leq 1$ for all indices i, each cycle of σ is of length 1 or 2. Given a permutation σ in R_n such that $\sigma_n = n-1$, we may thus deduce that $(n-1, n)$ is a cycle of σ. Therefore, $d_{n-1}(\text{trid}_n)$ is equal to:

$$\sum_{\substack{\sigma \in R_n \\ \sigma_n = n-1 \\ \sigma_{n-1} = n}} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n-2,\sigma_{n-2}} a_{n-1,n} a_{n,n-1} +$$
$$\sum_{\substack{\sigma \in R_n \\ \sigma_n = n}} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n-1,\sigma_{n-1}} a_{n,n}.$$

Given partitions μ and μ', we define the direct sum

$$\mu \oplus \mu' = \text{sort}\left(\mu \cdot \mu'\right)$$

of μ and μ' as the partition obtained by sorting the parts of μ and μ'. Now rewrite the above expression for $d_{n-1}(\text{trid}_n)$ as follows.

$$\alpha\gamma \sum_{\substack{\sigma \in R_n \\ \sigma_n = n-1 \\ \sigma_{n-1} = n}} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n-2,\sigma_{n-2}} + \beta \sum_{\substack{\sigma \in R_n \\ \sigma_n = n}} \chi^{(n-1,1)}_{\text{cycle}(\sigma)} a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n-1,\sigma_{n-1}}$$

$$= \alpha\gamma \sum_{\rho \in R_{n-2}} \chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2)} a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-2,\rho_{n-2}} + \beta \sum_{\rho \in R_{n-1}} \chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(1)} a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-1,\rho_{n-1}}$$

$$= \alpha\gamma \sum_{\rho \in R_{n-2}} \chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2)} a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-2,\rho_{n-2}} + \beta \sum_{\rho \in R_{n-1}} \left(\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(1)} - 2\chi^{(n-2,1)}_{\text{cycle}(\rho)}\right) a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-1,\rho_{n-1}} +$$
$$2\beta \cdot \text{PenImm}(\text{trid}_{n-1})$$

$$= \alpha\gamma \sum_{\rho \in R_{n-2}} \chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2)} a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-2,\rho_{n-2}} +$$
$$\alpha\beta\gamma \sum_{\rho \in R_{n-3}} \left(\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2,1)} - 2\chi^{(n-2,1)}_{\text{cycle}(\rho)\oplus(2)}\right) a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-3,\rho_{n-3}} +$$
$$\beta^2 \sum_{\rho \in R_{n-2}} \left(\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(1,1)} - 2\chi^{(n-2,1)}_{\text{cycle}(\rho)\oplus(1)}\right) a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-2,\rho_{n-2}} + 2\beta \cdot \text{PenImm}(\text{trid}_{n-1})$$

$$= \sum_{\rho \in R_{n-3}} \left(\alpha\beta\gamma\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2,1)} - 2\alpha\beta\gamma\chi^{(n-2,1)}_{\text{cycle}(\rho)\oplus(2)}\right) a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-3,\rho_{n-3}} +$$
$$\sum_{\rho \in R_{n-2}} \left(\beta^2\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(1,1)} - 2\beta^2\chi^{(n-2,1)}_{\text{cycle}(\rho)\oplus(1)} + \alpha\gamma\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2)} - (2\alpha\gamma - \beta^2)\chi^{(n-3,1)}_{\text{cycle}(\rho)}\right) a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-2,\rho_{n-2}} +$$
$$(2\alpha\gamma - \beta^2) \cdot \text{PenImm}(\text{trid}_{n-2}) + 2\beta \cdot \text{PenImm}(\text{trid}_{n-1})$$

$$= \sum_{\rho \in R_{n-4}} \left(\alpha\beta^2\gamma\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2,1,1)} - 2\alpha\beta^2\gamma\chi^{(n-2,1)}_{\text{cycle}(\rho)\oplus(2,1)} + (\alpha\gamma)^2\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2,2)} -\right.$$
$$\left.\alpha\gamma(2\alpha\gamma - \beta^2)\chi^{(n-3,1)}_{\text{cycle}(\rho)\oplus(2)} + (\alpha\gamma)^2\chi^{(n-5,1)}_{\text{cycle}(\rho)}\right) a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-4,\rho_{n-4}} +$$
$$\sum_{\rho \in R_{n-3}} \left(\beta^3\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(1,1,1)} - 2\beta^3\chi^{(n-2,1)}_{\text{cycle}(\rho)\oplus(1,1)} + \alpha\beta\gamma\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2,1)} - \beta(2\alpha\gamma - \beta^2)\chi^{(n-3,1)}_{\text{cycle}(\rho)\oplus(1)} +\right.$$
$$\left.\alpha\beta\gamma\chi^{(n-1,1)}_{\text{cycle}(\rho)\oplus(2,1)} - 2\alpha\beta\gamma\chi^{(n-2,1)}_{\text{cycle}(\rho)\oplus(2)} + 2\alpha\beta\gamma\chi^{(n-4,1)}_{\text{cycle}(\rho)}\right) a_{1,\rho_1} a_{2,\rho_2} \cdots a_{n-3,\rho_{n-3}} +$$
$$2\beta \cdot \text{PenImm}(\text{trid}_{n-1}) + \left(2\alpha\gamma - \beta^2\right) \text{PenImm}(\text{trid}_{n-2}) -$$
$$2\alpha\beta\gamma \cdot \text{PenImm}(\text{trid}_{n-3}) - (\alpha\gamma)^2 \text{PenImm}(\text{trid}_{n-4}).$$

Let $\rho \in R_{n-4}$, and suppose that ρ has exactly x fixed points. By the above lemma, we thus have that the character coefficient of the first sum above is equal to:

$$\alpha\beta^2\gamma(x+1) - 2\alpha\beta^2\gamma x + (\alpha\gamma)^2(x-1) -$$
$$\alpha\gamma(2\alpha\gamma - \beta^2)(x-1) + (\alpha\gamma)^2(x-1) = 0.$$

Let $\rho \in R_{n-3}$, and suppose that ρ has exactly y fixed points. By the above lemma, we thus have that the character coefficient of the latter sum above is equal to:

$$\beta^3(y+2) - 2\beta^3(y+1) + \alpha\beta\gamma y - \beta(2\alpha\gamma - \beta^2)y + \alpha\beta\gamma y -$$
$$2\alpha\beta\gamma(y-1) + 2\alpha\beta\gamma(y-1) = 0,$$

thus completing our proof. □

Let $a(n) = \text{A067331}(n)$ denote the n^{th} entry in the OEIS sequence A067331, with: $\text{A067331}(n) = \sum_{k=0}^{n} F_{k+1}F_{n+3-k}$. As indicated in [2], $a(n)$ is equal to the external path length of the Fibonacci tree of order $n+3$. The external path length of a tree is the sum of the levels of its leaves. It is easily seen that:

$$a(n) = 2a(n-1) + a(n-2) - 2a(n-3) - a(n-4)$$

with $a(0) = 2$, $a(1) = 5$, $a(2) = 12$, $a(3) = 25$. By Lemma 2.1, the penultimate immanant of $\mathrm{trid}_n = \mathrm{trid}_n^{1,1,1}$ is equal to

$$\mathrm{PenImm}(\mathrm{trid}_n) = 2 \cdot \mathrm{PenImm}(\mathrm{trid}_{n-1}) + \mathrm{PenImm}(\mathrm{trid}_{n-2}) -$$

$$2 \cdot \mathrm{PenImm}(\mathrm{trid}_{n-3}) - \mathrm{PenImm}(\mathrm{trid}_{n-4})$$

with $\mathrm{PenImm}(\mathrm{trid}_3) = 2$, $\mathrm{PenImm}(\mathrm{trid}_4) = 5$, $\mathrm{PenImm}(\mathrm{trid}_5) = 12$, and $\mathrm{PenImm}(\mathrm{trid}_6) = 25$. We thus have that

$$\mathrm{A067331}(n) = \mathrm{PenImm}(\mathrm{trid}_{n+3}^{1,1,1})$$

for all $n \in \mathbb{N}_0$, and we thus have that

$$\mathrm{EPL}(T_n) = \mathrm{PenImm}(\mathrm{trid}_n^{1,1,1})$$

for $n \geq 3$ as desired.

3 A representation-theoretic proof of a new binomial identity

In this section, we prove (1.3) using (1.2). From (1.1) we have that

$$\mathrm{EPL}(T_n) = \sum_{\sigma} \chi^{(n-1,1)}(\sigma),$$

where the above sum is as given in (1.2), i.e. the above sum is over all permutations σ in S_n such that $|\sigma_i - i| \leq 1$ for all indices i. It is easily seen that given a permutation σ of this form, the cycle type of σ is of the form $(2^i, 1^j) \vdash n$. Conversely, suppose that $\mu \vdash n$ is a partition of the form $(2^i, 1^j)$. It seems natural to ask: how many permutations σ satisfying $\forall i \;\; |\sigma_i - i| \leq 1$ are of cycle type μ? Observe that a 2-cycle of a permutation σ of this form must be of the form $(j, j+1)$. But then it is easily seen that that there are precisely

$$\binom{\ell(\mu)}{(\mu^*)_2}$$

permutations of this form, letting μ^* denote the conjugate of μ, thus proving (1.3). Note that we are using the standard convention whereby $\lambda_i = 0$ for an arbitrary partition λ and an arbitrary index i satisfying $i > \ell(\lambda)$. So, for a partition μ satisfying $\ell(\mu^*) \leq 2$, we have that $(\mu^*)_2$ is precisely the number of entries of μ which are equal to 2.

It now seems natural to make use of Claim 1.2. In particular, from (1.3) together with Claim 1.2, we thus arrive at the following beautiful combinatorial formula:

$$\mathrm{EPL}(T_n) = \sum_{\substack{\mu \vdash n \\ \ell(\mu^*) \leq 2}} \binom{\ell(\mu)}{(\mu^*)_2} \left(\ell(\mu) - (\mu^*)_2 - 1\right) \tag{3.1}$$

The elegant binomial identity

$$\mathrm{EPL}(T_n) = \sum_{i=0}^{n-3} F_{i+1} F_{n-i} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (n - 2i - 1)$$

follows immediately from (3.1) by rewriting the sum given in (3.1) as a sum over the possible number of 2's in a partition $\mu \vdash n$ such that $\text{diag}(\mu)$ consists of at most 2 columns.

We remark that our technique which was used to prove the binomial identity

$$\text{EPL}(T_n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (n-2i-1)$$

may be applied to immanants more generally. In particular, we have that:

$$\text{Imm}^{\lambda}(\text{trid}_n^{1,1,1}) = \sum_{\substack{\mu \vdash n \\ \ell(\mu^*) \leq 2}} \binom{\ell(\mu)}{(\mu^*)_2} \chi_{\mu}^{\lambda}. \tag{3.2}$$

Letting $\lambda = (1^n)$ with respect to (3.2), we thus arrive at the following known binomial identity:

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (-1)^i = \frac{1}{2} \left((-1)^{\lfloor \frac{n}{3} \rfloor} + (-1)^{\lfloor \frac{n+1}{3} \rfloor} \right).$$

Letting $\lambda = (n)$ with respect to (3.2), we thus arrive at the following well-known binomial identity:

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

4 Conclusion

We currently leave it as an open problem to construct a bijective, as opposed to recursive, proof of (1.1), and we leave it as an open problem to construct a formula for evaluating an arbitrary hook immanant of an arbitrary tridiagonal matrix.

References

[1] Bodeen, J., Butler, S., Kim, T., Sun, X., and Wang, S. (2014). Tiling a strip with triangles. *Electron. J. Combin.* **21.1** #P1.7.

[2] OEIS Foundation Inc. (2011). The On-Line Encyclopedia of Integer Sequences.