Representing a sequence as $[x^n]\,G(x)^n$

Peter Bala, August 2015

Let $a(n)$ be an integer sequence with $a(0) = 1$. We show that there exists a formal power series $G(x)$, having rational coefficients, such that $a(n) = [x^n]\,G(x)^n$ for $n \ge 0$. This result can be found (in a slightly disguised form) in [Stanley, Enumerative Combinatorics, Vol. 2 - see Exercise 5.56 (a), p. 98 and its solutions on pp. 146-147].

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We will need the following result, which is the particular case $k = 0$ of part (ii) of 2.1 Theorem in [1].

Proposition 1. *Let $G(t)$ be any element of $\mathbb{C}[[t]]$. Then the equation $f(x) = xG(f(x))$ has a unique solution in $\mathbb{C}[[x]]$ and*

$$[x^n]\frac{1}{1 - xG'(f(x))} \quad = \quad [t^n]G(t)^n. \tag{1}$$

□

Proposition 2. *Let $a(n)$ be an integer sequence with $a(0) = 1$. Then there exists a formal power series $G(x)$ with rational coefficients such that*

$$a(n) \quad = \quad [x^n]\,G(x)^n \quad \text{for all } n \ge 0.$$

Proof. Define the power series $f(x)$ by

$$f(x) \quad = \quad x\exp\left(\sum_{n\ge1} a(n)\frac{x^n}{n}\right). \tag{2}$$

Clearly, the expansion of $f(x)$ will be a power series in x with rational coefficients.

Logarithmically differentiating (2) gives

$$\frac{xf'(x)}{f(x)} = \sum_{n\ge0} a_n x^n \tag{3}$$

so that

$$a(n) \quad = \quad [x^n]\frac{xf'(x)}{f(x)}. \tag{4}$$

Define a power series $G(x)$ by

$$G(x) = \frac{x}{\bar{f}(x)}, \tag{5}$$

where $\bar{f}(x)$ denotes the compositional inverse of $f(x)$. The function $G(x)$ when expanded as a power series in x will have rational coefficients.

It follows from (5) that $f(x)$ satisfies the functional equation

$$f(x) = xG(f(x)). \tag{6}$$

Thus $G(x)$ and $f(x)$ satisfy the conditions of Proposition 1. In order to apply the proposition we need to calculate $1/(1 - xG^{'}(f(x)))$.

Differentiating (6) with respect to x gives

$$f^{'}(x) \quad = \quad xG^{'}(f(x))f^{'}(x) + G(f(x)).$$

Hence

$$\begin{aligned} 1 - xG^{'}(f(x)) &= \frac{G(f(x))}{f'(x)} \\ &= \frac{f(x)}{xf'(x)}, \end{aligned}$$

by (6).

Thus

$$\frac{1}{1 - xG'(f(x))} = \frac{xf^{'}(x)}{f(x)}. \tag{7}$$

Using (7), Proposition 1 yields

$$\begin{aligned} [t^n]G(t)^n &= [x^n]\frac{xf^{'}(x)}{f(x)} \\ &= a(n), \end{aligned}$$

by (4). $\square$

Remarks.

1) In Proposition 2, the power series $G(x)$ will be integral iff the power series $f(x)$ determined by (2) is integral.

2) The expansion of $G(x)$ begins

$$G(x) = 1 + a(1)x + \left(a(2) - a(1)^2\right)\frac{x^2}{2!} + \left(2a(3) - 6a(1)a(2) + 4a(1)^3\right)\frac{x^3}{3!} + \cdots.$$

A simple induction argument shows that the n-th coefficient in this expansion is a polynomial in $a(1), ..., a(n)$.

3) For $m \neq 0$ the power series $G_m(x) := \frac{1}{m}G(mx)$ also has the property that $a(n) = [x^n]\, G_m(x)^n$. The power series $G(x)$ defined by (5) is the unique power series having the properties $G(0) = 1$ and $a(n) = [x^n]\, G(x)^n$ for $n \geq 0$.

Example. Let P and Q be integers. The Lucas sequence of the second kind $V_n \equiv V_n(P, Q)$ is the integer sequence

$$V_n = a^n + b^n,$$

where

$$a = \frac{1}{2}\left(P + \sqrt{D}\right)$$
$$b = \frac{1}{2}\left(P - \sqrt{D}\right)$$

and

$$D = P^2 - 4Q.$$

Examples include the Lucas Numbers $V_n(1, -1)$ =A000032, the Pell-Lucas numbers $V_n(2, -1)$ = A002203 and the Jacobsthal-Lucas numbers $V_n(1, -2)$ = A014551.

A simple calculation gives

$$f(x) := x\exp\left(\sum_{n\geq 1} V_n \frac{x^n}{n}\right).$$
$$= \frac{x}{1 - Px + Qx^2}.$$

We then find

$$G(x) := \frac{x}{\bar{f}(x)},$$
$$= \frac{1 + Px + \sqrt{1 + 2Px + (P^2 - 4Q)\, x^2}}{2}.$$

Hence, by Proposition 2

$$V_n \quad = [x^n] \quad \left(\frac{1 + Px + \sqrt{1 + 2Px + (P^2 - 4Q)\, x^2}}{2} \right)^n \text{ for } n \geq 1.$$

For instance, taking $P = 1, Q = -1$ we have

$$\text{Lucas}(n) \quad = [x^n] \quad \left(\frac{1 + x + \sqrt{1 + 2x + 5x^2}}{2} \right)^n \text{ for } n \geq 1.$$

REFERENCES

[1] I. M. Gessel, A Factorization for Formal Laurent Series and Lattice Path Enumeration, J. of Combinatorial Theory, Series A 28, 321-337 (1980) online at http://people.brandeis.edu/~gessel/homepage/papers/factorization.pdf

[2] Wikipedia, Lucas Sequence