NO COMMON TERMS IN THE SEQUENCES $\sigma(p^i)$ AND $\sigma(p^{i+1})$ AS p RUNS THROUGH THE PRIMES.

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Abstract. The sum-of-divisors function has distinct values when computed for a power of a prime and a next-higher power of a prime.

1. Sequences of the form $\sum_{i=0}^{e} p^i$

The well-known formula for the multiplicative function σ [1, A000203] is based on

$$\sigma(p^e) = \frac{p^{e+1}-1}{p-1} = 1 + p + p^2 + \cdots + p^e \tag{1}$$

for prime powers p^e. The associated sequences are A008864, A060800 and A131991–A131993 for $p = 1 \ldots 5$ as p runs through the primes.

It is easy to see (see the comment in A008864) that the sequence $\sigma(p)$ has no common terms with the sequences $\sigma(p^e)$ for any exponent $e \geq 2$. We show that also sequences with exponents that differ by one have no common terms, which means

$$\sigma(p^e) \neq \sigma(q^{e+1}), \forall p, q(prime). \tag{2}$$

Proof. The equation requires solution of the diophantine equation

$$\sigma(p^e) = \sigma(q^{e+1}); \tag{3}$$

$$1 + p + p^2 + \cdots p^e = 1 + q + q^2 + \cdots + q^{e-1} \tag{4}$$

for two primes p and q, where $p > q$ because the right hand side has one more term. Subtracing one and factoring yielss

$$p(1 + p + p^2 + \cdots p^{e-1}) = q(1 + q + q^2 + \cdots + q^e) \tag{5}$$

Working modulo p or q means

$$0 \equiv q(1 + q + q^2 + \cdots + q^e) \pmod{p}; \tag{6}$$

$$0 \equiv p(1 + p + p^2 + \cdots + p^{e-1}) \pmod{q} \tag{7}$$

and because neither $0 \equiv q \pmod{p}$ nor $0 \equiv p \pmod{q}$ also

$$0 \equiv 1 + q + q^2 + \cdots + q^e \pmod{p}; \tag{8}$$

$$0 \equiv 1 + p + p^2 + \cdots + p^{e-1} \pmod{q}. \tag{9}$$

This requires

$$1 + q + q^2 + \cdots + q^e = \alpha p \wedge 1 + p + p^2 + \cdots + p^{e-1} = \beta q \tag{10}$$

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with two integers α and β. Insertion into (5) yields $\alpha = \beta$, so

$$1 + q + q^2 + \cdots + q^e = \alpha p; \tag{11}$$

$$1 + p + p^2 + \cdots + p^{e-1} = \alpha q. \tag{12}$$

Subtracting both equations means

$$p - q + (p^2 - q^2) + \cdots + (p^e - q^e) + (p^{e-1} - q^{e-1}) - q^e = -\alpha(p - q). \tag{13}$$

All but the last terms on the left hand side are multiples of $p-q$, so working modulo $p - q$ requires $q^e \equiv 0 \pmod{p-q}$, so

$$q^e = k(p - q) \tag{14}$$

for some integer k. Due to the uniqueness of the prime factorization of q^e, the right hand side must split the factors as

$$k = q^{e-\delta} \wedge p - q = q^\delta \tag{15}$$

for some integer $0 \le \delta \le e$. There are two cases

(1) p and q are odd primes, so the left hand side $p - q$ is even and the right hand side q^δ must also be even, which leads to a contradiction because then q must be even.

(2) $p = 3$ and $q = 2$. This special case is with (2) equivalent to

$$\frac{3^{e+1} - 1}{3 - 1} = \frac{2^{e+2} - 1}{2 - 1}; \tag{16}$$

$$3^{e+1} - 1 = 2(2^{e+2} - 1). \tag{17}$$

This has no solution as may be checked explicitly for small e and follows then because the left hand side grows like a power of 3 and the right hand side only with a power of 2.

□

In summary, A008864 has no common term with A060800; A060800 has no common term with A131991 and so on.

References

1. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://oeis.org/. MR 1992789

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