A triangle for calculating A060627.

Peter Bala, April 25, 2017

Oste and van der Jeugt [1, Section 7] show that a continued fraction of the form

$$\cfrac{1}{1 - xd_0 - \cfrac{xh_1}{1 - xd_1 - \cfrac{xh_2}{1 - xd_2 - \cfrac{xh_3}{1 - xd_3 - \cdots}}}} \tag{1}$$

is the generating function for 2-Motzkin paths weighted by the integers d_i and h_i. This combinatorial interpretation allows one to rapidly calculate the terms of a sequence whose generating function can be expressed as a continued fraction of the form (1). The results are conveniently displayed in the form of a lower triangular array, where the $d's$ occur as multiplication factors along diagonals of the array and the $h's$ as horizontal multiplication factors along rows of the array. In the particular case of A060627, which gives the expansion of the Jacobian elliptic function $\operatorname{cn}(x, k)$, the bivariate generating function can be expressed as the continued fraction

$$1/(1 - x/(1 - 4k^2x/(1 - 3^2x/(1 - 16k^2x/(1 - 5^2x/(1 - ...)))))).$$

So in this case the $d's$ are all zero and the horizontal multiplication factors are given by the formula $h_{2n+1} = (2n + 1)^2, h_{2n} = (2nk)^2$. The row polynomials of A060627 appear on the leading diagonal of the following lower triangular array:

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
\downarrow & & & & & & & & \\
1 & \text{— x1—>} & 1 & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
1 & \text{— x}4k^2\text{—>} & 1+4k^2 & \text{—x1—>} & 1+4k^2 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
1 & \text{— x9—>} & 10+4k^2 & \text{—x}4k^2\text{—>} & 1+44k^2+16k^4 & \text{—x1—>} & 1+44k^2+16k^4 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \ddots
\end{array}$$

References

[1] R. Oste and J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, Electronic Journal of Combinatorics 22(2) (2015), #P2.8. Section 7