Meixner polynomials and Brouncker's continued fractions for Pi

Peter Bala, April 09 2024

Lord Brouncker found an infinite family of continued fractions for Pi that includes the following particular cases [Osler, equation 4]: for n >= 0 there holds

1^2/(8*n+6 + 3^2/(8*n+6 + 5^2/(8*n+6 + 7^2/(8*n+6 + ...))))

= ((2*n + 1)!!/(2^n*n!))^2 * Pi - (4*n + 3).

The purpose of this note is to express the continued fraction as a series: we sketch a proof that for n >= 0,

1^2/(8*n+6 + 3^2/(8*n+6 + 5^2/(8*n+6 + 7^2/(8*n+6 + ...))))

= 4*((2*n + 1)!!)^4 * Sum_{i >= 1} (-1)^(i+1)/((2*i + 1)*

R(2*n+1, 2*i)*R(2*n+1, 2*i+2)),

where R(n, x) denotes the n-th row polynomial of triangle A060524. There is a well-known connection between continued fractions and orthogonal polynomials so this result may be known, but we were unable to find a reference.

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The row polynomials of A060524 are particular examples of Meixner polynomials and have the e.g.f.

F(t, x) = (1 + t)^((x-1)/2)/(1 - t)^((x+1)/2)

= Sum_{n >= 0} R(n, x)*t^n/n! ... (1)

= 1 + x*t + (1 + x^2)^t^2/2! + (5*x + x^3)*t^3/3!

\+ (9 + 14*x*2 + x^4)*t^4/4! + ...

It is easily verified that the g.f. F(t, x) satisfies

4*t*d/dt(F(t, x)) + 2*F(t, x)

= (x + 1)*F(t, x + 2) - (x -1)*F(t, x - 2)

which leads to the identity for the row polynomials

(4*n + 2)*R(n, x) = (x + 1)*R(n, x+2) - (x - 1)*R(n, x-2) ... (2)

For each n >= 0, we define a sequence {B_n(k): k >= 0} by setting

B_n(k) = (2*k + 1)!! * R(2*n+1, 2*k+2).

Using (2), it is easy to check that B_n(k) satisfies the 3-term recurrence

B_n(k) = (8*n + 6)*B_n(k-1) + (2*k - 1)^2*B_n(k-2) ... (3)

for k > = 2.

The initial conditions

B_n(0) = 2*(2*n + 1)!!^2, B_n(1) = 2*(2*n + 1)!!^2 *(8*n+6) ...(4)

are obtained from the e.g.f. (1) and making use of (2).

For each n >= 0, we define a second sequence {A_n(k): k >= 0} by

A_n(k) = B_n(k) * Sum_{i = 1..k} (-1)^(i+1)/((2*i + 1)*

R(2*n+1, 2*i)*R(2*n+1, 2*i+2)),

so that

lim_{k -> oo} A_n(k)/B_n(k) =

Sum_{i >= 1} (-1)^(i+1)/((2*i + 1)*R(2*n+1, 2*i)*R(2*n+1, 2*i+2))

... (5)

With a little bit more work one can verify that A_n(k) satisfies the same 3-term recurrence as B(n,k):

A_n(k) = (8*n + 6)*A_n(k-1) + (2*k - 1)^2*A_n(k-2) ... (6)

for k >= 2. The initial conditions

A_n(0) = 0, A_n(1) = 1/(2*(2*n + 1)!!^2) ... (7)

are obtained from the e.g.f. (1) and making use of (2).

We scale the sequences {A_n(k)} and {B_n(k)} by defining

A_n(k) = (2*(2*n + 1)!!^2) * A_n(k)

and

B_n(k) = B_n(k)/(2*(2*n + 1)!!^2).

The sequences {**A**_n(k): k >= 0} and {**B**_n(k): k >= 0} satisfy the same 3-term recurrences as {A_n(k) : k >= 0} and {B_n(k) : k >= 0} From (4) and (7) the initial conditions now become

A_n(0) = 0; **A**_n(1) = 1;

B_n(0) = 1; **B**_n(1) = 8*n + 6.

By comparison with the fundamental 3-term recurrences satisfied by the numerators and denominators of the convergents to a generalised continued fraction, we find that for n >= 0, k >= 1,

4*((2*n + 1)!!)^4 * A_n(k)/B_n(k) = **A**_n(k)/**B**_n(k) =

1^2/(8*n+6 + 3^2/(8*n+6 + 5^2/(8*n+6 + ... + (2*k-1)^2/(8*n+6)))).

Letting k -> oo, and using (5), gives

4*((2*n + 1)!!)^4 * lim_{k -> oo} A_n(k)/B_n(k)

= 4*((2*n + 1)!!)^4 * Sum_{k >= 1} (-1)^(k+1)/((2*k + 1)*
R(2*n+1, 2*k)*R(2*n+1, 2*k+2))

= 1^2/(8*n+6 + 3^2/(8*n+6 + 5^2/(8*n+6 + ... + (2*k - 1)^2/(8*n
+6 + ...)))).

References

T. J. Osler, The missing fractions in Brouncker's sequence of continued fractions for Pi, The Mathematical Gazette, 96(2012), pp. 221-225